UNIONS OF CHAINS OF PRIMES

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Abstract. The union of an ascending chain of prime ideals is not always prime. The union of an ascending chain of semi-prime ideals is not always semi-prime. We show that these two properties are independent. We also show that the number of non-prime unions of subchains in a chain of primes in a PI-algebra does not exceed the PI-class minus one, and this bound is tight.

1. Introduction

In a commutative ring, the union of a chain of prime ideals is prime, and the union of a chain of semiprime ideals is semiprime. This paper demonstrates and measures the failure of these chain conditions in general.

Definition 1.1. *A ring has the* **(semi)prime chain property** *(denoted $\mathcal{P}^{\uparrow}$ and $\mathcal{SP}^{\uparrow}$, respectively) if the union of any countable chain of (semi)prime ideals is always (semi)prime.*[1]

The property $\mathcal{SP}^{\uparrow}$ was recognized by Fisher and Snider [4] as the missing hypothesis for Kaplansky's conjecture on regular rings, and they gave an example of a ring without $\mathcal{SP}^{\uparrow}$.

Our focus is on $\mathcal{P}^{\uparrow}$. The class of rings satisfying $\mathcal{P}^{\uparrow}$ is quite large. An easy exercise shows that every commutative ring satisfies $\mathcal{P}^{\uparrow}$, and the same argument yields that the union of strongly prime ideals is strongly prime ($P \triangleleft R$ is strongly prime if R/P is a domain). In fact, we have the following result:

Proposition 1.2. *Every ring R which is a finite module over a central subring, satisfies $\mathcal{P}^{\uparrow}$.*

Proof. Write $R = \sum_{i=1}^{t} Cr_i$ where $C \subseteq \mathrm{Cent}(R)$. Suppose $P_1 \subset P_2 \subset \cdots$ is a chain of prime ideals, with $P = \cup P_i$. If $a, b \in R$ with

$$\sum Car_ib = \sum aCr_ib = aRb \subseteq P,$$

then there is n such that $ar_ib \in P_n$ for $1 \leq i \leq t$, implying $aRb = \sum Car_ib \subseteq P_n$, and thus $a \in P_n$ or $b \in P_n$. $\square$

(For a recent treatment of the correspondence of infinite chains of primes between a ring R and a central subring, see [12]).

The class of rings satisfying $\mathcal{P}^{\uparrow}$ also contains every ring that satisfies ACC (ascending chain condition) on primes, and is closed under homomorphic images and

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[1]For simplicity we deal only with countable chains throughout the paper, but the arguments are general.

central localizations. This led some mathematicians to believe that it holds in general. On the other hand, Bergman produced an example lacking $\mathcal{P}^{\uparrow}$ (see Example 2.1 below), implying that the free algebra on two generators does not have $\mathcal{P}^{\uparrow}$.

Obviously, the property $\mathcal{P}^{\uparrow}$ follows from the maximum property on families of primes. On the other hand, $\mathcal{P}^{\uparrow}$ implies (by Zorn's lemma) the following maximum property: for every prime Q contained in any ideal I, there is a prime P maximal with respect to $Q \subseteq P \subseteq I$.

In Section 4 we show that $\mathcal{P}^{\uparrow}$ and $\mathcal{SP}^{\uparrow}$ are independent, by presenting an example (due to Kaplansky and Lanski) of a ring satisfying $\mathcal{P}^{\uparrow}$ and not $\mathcal{SP}^{\uparrow}$, and an example of a ring satisfying $\mathcal{SP}^{\uparrow}$ but not $\mathcal{P}^{\uparrow}$.

We say that an ideal is **union-prime** if it is a union of a chain of primes, but is not itself prime. (If $\{P_\lambda\}$ is an ascending chain of primes, then $R/\bigcup P_\lambda = \lim_{\rightarrow} R/P_\lambda$ is a direct limit of prime rings). The **$\mathcal{P}^{\uparrow}$-index** of the ring R is the maximal number of non-prime unions of subchains of a chain of prime ideals in R (or infinity if the number is unbounded, see Proposition 5.3). Section 2 extends Bergman's example by showing that the $\mathcal{P}^{\uparrow}$-index of the free (countable) algebra is infinity. A variation of this construction, based on free products, is presented in Section 3. After defining the $\mathcal{P}^{\uparrow}$-index in Section 5, in Section 6 we discuss PI-rings, showing that the $\mathcal{P}^{\uparrow}$-index does not exceed the PI-class minus one, and this bound is tight. We thank the anonymous referee for careful comments on a previous version of this paper.

2. Monomial algebras

Fix a field F. We show that $\mathcal{P}^{\uparrow}$ and $\mathcal{SP}^{\uparrow}$ fail in the free algebra (over F) by constructing an (ascending) chain of primitive ideals whose union is not semiprime. Let us start with a simpler theme, whose variations have extra properties.

Example 2.1 (A chain of prime ideals with non-semiprime union). *Let R be the free algebra in the (noncommuting) variables x, y. For each n, let*

$$P_n = \langle xx, xyx, xy^2x, \dots, xy^{n-1}x \rangle.$$

As a monomial ideal, it is enough to check primality on monomials. If $uRu' \subseteq P_n$ for some words u, u', then in particular $uy^nu' \in P_n$, which forces a subword of the form xy^ix (with $i < n$) in u or in u'; hence either $u \in P_n$ or $u' \in P_n$.

On the other hand $\bigcup P_n = (RxR)^2$ which is not semiprime.

This example, due to G. Bergman, appears in [9, Exmpl. 4.2]. Interestingly, primeness is always maintained in the following sense ([9, Lem. 4.1], also due to Bergman): for every countable chain of primes $P_1 \subset P_2 \subset \cdots$ in a ring R, the union $\bigcup(P_n[[\zeta]])$ is a prime ideal of the power series ring $R[[\zeta]]$.

Since in Example 2.1 $\bigcup P_n = (RxR)^2$, if $Q \lhd R$ is a prime containing the union then $x \in Q$ so R/Q is commutative. In particular, a chain of prime ideals starting from the chain $P_1 \subset P_2 \subset \cdots$ has only one union-prime. Let us exhibit a (countable) chain providing infinitely many union-primes.

Example 2.2 (A prime chain with infinitely many union-primes). *Let R be the free algebra generated by x, y, z. For a monomial w we denote by $\deg_y w$ the degree of w with respect to y. For $i, n \geq 1$, consider the monomial ideals*

$$I_{i,n} = RxxR + RxzxR + \cdots + Rxz^{i-1}xR + \sum_{\deg_y w < n} Rxz^ixwxz^ixR,$$

which form an ascending chain with respect to the lexicographic order on the indices (i,n), since $xz^ix \in I_{i',n}$ for every $i' > i$. To show that $I_{i,n}$ are prime, suppose that u, u' are monomials such that $u, u' \notin I_{i,n}$ but $uRu' \subseteq I_{i,n}$. Then $uz^iy^nz^iu' \in I_{i,n}$. Since none of the monomials $xz^{i'}x$ ($i' < i$) is a subword of u or u', they are not subwords of $uz^iy^nz^iu'$, forcing $uz^iy^nz^iu'$ to have a subword of the form xz^ixwxz^ix where $\deg_y w < n$. It follows that $z^iy^nz^i$ is a subword of z^ixwxz^i, contrary to the degree assumption. Now, for every i,

$$\bigcup_n I_{i,n} = RxxR + RxzxR + \dots + Rxz^{i-1}xR + (Rxz^ixR)^2,$$

which contains $(Rxz^ixR)^2$ but not Rxz^ixR, so it is not semiprime.

In particular the $\mathcal{P}^{\uparrow}$-index of R (see Proposition 5.3) is infinity. In Section 6 we show that this phenomenon is impossible in PI algebras: there, the number of union-primes in a prime chain is bounded by the PI-class.

Remark 2.3. *The ideals P_n in Example 2.1 are in fact primitive. Indeed, Bell and Colak* [1] *proved that any finitely presented prime monomial algebra is either primitive or PI (also see* [8]*), and R/P_n contains a free subalgebra, e.g. $k\langle xy^n, xy^{n+1}, \dots\rangle$.*

The same effect can be achieved by using idempotents.

Example 2.4 (A chain of primitive ideals with non-semiprime union)**.** *Let R be the free algebra in the variables e, y, modulo the relation $e^2 = e$. Every monomial has a unique shortest presentation as a word (replacing e^2 by e throughout). Ordering monomials first by length and then lexicographically, every element f has an upper monomial $\bar{f}$. Notice that $\overline{fy^ng} = \bar{f}y^n\bar{g}$.*

For each n, let

$$P_n = \langle eye, ey^2e, \dots, ey^{n-1}e\rangle.$$

To show that P_n is a prime ideal, assume that $fy^ng \in P_n$. Then $\bar{f}y^n\bar{g} = \overline{fy^ng} \in P_n$, forcing $\bar{f} \in P_n$ or $\bar{g} \in P_n$ as in Example 2.1. The claim follows by induction on the number of monomials.

To show that the ideal P_n is primitive, it is enough by [7] *to prove that $e(R/P_n)e$ is a primitive ring. We construct an isomorphism between $e(R/P_n)e$ and the countably generated free algebra $F\langle z_0, z_1, \dots\rangle$ by sending ey^me for $m \geq n$ (which clearly generate a free algebra) to z_{m-n}. But the free algebra is primitive (see* [6, Prop. 11.23]*).*

On the other hand $\bigcup P_n = ReyReR = ReRyeR$, which contains $(ReyR)^2$ but not ey, so is not semiprime.

Remark 2.5. *We say that a ring is* **uniquely-$\mathcal{P}^{\uparrow}$** *if it has a unique minimal prime over every chain of prime ideals. Since the intersection of a descending chain of primes is prime, Zorn's lemma shows that there are minimal primes over every ideal, in particular over every union-prime.*

In the topology of the spectrum, a net $\{P_\lambda\}_{\lambda\in\Lambda}$ of primes converges to a prime Q if and only if $\bigcap_{\lambda\in\Lambda}\bigcup_{\lambda'\geq\lambda} P_{\lambda'} \subseteq Q$; in particular when $\{P_\lambda\}$ is an ascending chain, $\lim P_\lambda = Q$ if and only if $\bigcup P_\lambda \subseteq Q$. Therefore, the spectrum can identify minimal primes over union-primes. It seems that the spectrum cannot distinguish $\mathcal{P}^{\uparrow}$ from uniquely-$\mathcal{P}^{\uparrow}$.

In the examples of this section, there is a unique minimal prime over every union-prime. In Example 4.5 below the situation is different: the union-prime ideal constructed there is the intersection of two primes containing it.

3. Prime ideals in free products

In Example 2.1 there are infinitely many (incomparable) prime ideals lying over the chain. We modify this example, in order to obtain a chain over which there is unique prime. In Example 2.1 we considered ideals of the free algebra, which can be written as a free product $F[x] *_F F[y]$. The quotient over the radical of the union over the chain is the "second" component $F[y]$, which we would like to replace by the field $F(y)$. The proof that the ideals are prime is somewhat delicate; we thank the referee for pointing this out.

Let F be a field and let A, B be F-algebras, with given vector space decompositions $A = F \oplus A_0$ and $B = F \oplus B_0$. The free product $A *_F B$ can be viewed as the tensor algebra $T(A_0 \oplus B_0) = F \oplus \bigoplus_{n \geq 1} (A_0 \oplus B_0)^{\otimes n}$, modulo the relations $a \otimes a' = aa'$ and $b \otimes b' = bb'$ for every $a, a' \in A$ and $b, b' \in B$. We will omit the tensor symbol.

Fixing the decomposition $F[x] = F \oplus xF[x]$ and an arbitrary decomposition $B = F \oplus B_0$, we consider ideals of the free product $R = F[x] *_F B$. The tensor algebra is graded by x, once we declare that $\deg(b) = 0$ for every $b \in B_0$, and this grading induces a grading on R.

Let $W \subseteq B$ be a vector space containing F. We say that W is **restricted** if for every finite dimensional subspace $V \subseteq B$ there is an element $b \in B$ such that $Vb \subseteq B_0$ and $Vb \not\subseteq W$; and an element $b' \in B$ such that $b'V \subseteq B_0$ and $b'V \not\subseteq W$.

Theorem 3.1. *The ideal $P = RxWxR$ of R is prime whenever $W \subseteq B$ is a restricted subspace.*

Proof. Write $W = F \oplus W_0$ where $W_0 = W \cap B_0$. Let L' be the ideal of R generated by x^2. For $n \geq 0$ let us denote the vector spaces

$$L_n = BxB_0xB_0 \cdots xB_0xB,$$

where the degree with respect to x is n; so that $L_0 = B$ and $L_1 = BxB$. Setting $L = \sum_{n \geq 0} L_n$, we have that $R = L' \oplus L$.

Let $P_n = L_n \cap P$; so $P_0 = P_1 = 0$, and for $n \geq 2$,

$$P_n = \sum BxB_0x \cdots xB_0xW_0xB_0x \cdots xB_0xB$$

where in each summand one of the intermediate entries is W_0 and all the others are equal to B_0. For example $P_2 = BxW_0xB$ and $P_3 = BxW_0xB_0xB + BxB_0xW_0xB$. Now, since $F \subseteq W$, we have that $L' = RxxR = RxFxR \subseteq RxWxR = P$, and we can compute:

$$\begin{aligned}
P &= L' + P \\
&= L' + (L' + L)xWx(L' + L) \\
&= L' + LxWxL \\
&= L' + LxW_0xL \\
&= L' + \sum_{d,d' \geq 0} L_d x W_0 x L_{d'} \\
&= L' + \sum_{n \geq 0} \left(\sum_{d+d'=n} L_d x W_0 x L_{d'} \right) \\
&= L' + \sum_{n \geq 2} P_n,
\end{aligned}$$

since modulo L', $xBx \equiv xB_0x$ and $xWx \equiv xW_0x$.

Let $m \geq 1$. As a vector space, $L_m \cong B \otimes B_0 \otimes \cdots \otimes B_0 \otimes B$ with $m+1$ factors. This isomorphism carries P_m to $\sum B \otimes B_0 \otimes \cdots \otimes B_0 \otimes W_0 \otimes B_0 \otimes \cdots \otimes B_0 \otimes B$ as above, and there is an isomorphism

$$\psi_m : L_m/P_m \longrightarrow B \otimes \overline{B_0} \otimes \cdots \otimes \overline{B_0} \otimes B$$

with $m-1$ factors of the form $\overline{B_0} = B_0/W_0$. The image of $g \in L_m$ in $\overline{L_m} = L_m/P_m$ will be denoted by $\overline{g}$, hoping that no confusion is incurred by the double usage of the over-line.

We need to show that P is prime. Since $L' \subseteq P$, it suffices to show that if $f, f' \in L$ and $f, f' \notin P$, then $fBf' \not\subseteq P$. Furthermore since R is graded with respect to x, and P is a homogeneous ideal with respect to this grading, we may assume that f, f' are homogeneous with respect to x, so we can write

$$f = \sum_i a_{0,i} x a_{1,i} x \cdots x a_{n,i} \in L_n$$

and

$$f' = \sum_j a'_{0,j} x a'_{1,j} x \cdots x a'_{n',j} \in L_{n'}$$

where $a_{0,i}, a_{n,i}, a'_{0,j}, a'_{n',j} \in B$ and $a_{t,i}, a'_{t',j} \in B_0$ for $0 < t < n$ and $0 < t' < n'$. Let V be the vector space spanned by all the $a_{n,i}$ and V' the vector space spanned by all the $a'_{0,j}$. We say that f "ends in V" and f' "begins in V'". By assumption, there are elements $b, b' \in B$ such that $Vb, b'V' \subseteq B_0$, while $Vb \not\subseteq W_0$ and $b'V' \not\subseteq W_0$.

Since $Vb, b'V' \subseteq B_0$, we have that $fbxb'f' \in L_{n+n'+1}$. Consider the commutative diagram

$$\begin{tikzcd}
L_n \otimes L_{n'} \arrow[r, "\theta"] \arrow[d, "m"] & \overline{L_n} \otimes \overline{L_{n'}} \arrow[r, "\psi_n \otimes \psi_{n'}"] \arrow[d, "\bar{m}"] & (B \otimes \overline{B_0} \otimes \cdots \otimes \overline{B_0} \otimes B) \otimes (B \otimes \overline{B_0} \otimes \cdots \otimes \overline{B_0} \otimes B) \arrow[d] \\
L_{n+n'+1} \arrow[r] & \overline{L_{n+n'+1}} \arrow[r, "\psi_{n+n'+1}"] & B \otimes \overline{B_0} \otimes \cdots \otimes \overline{B_0} \otimes B
\end{tikzcd}$$

where the domain of definition of the top-to-bottom maps is the elements in $L_n \otimes L_{n'}$ such that the left factor ends in B_0 and the right factor begins in B_0 (and not merely in B), and their image. Here, $m(g \otimes g') = gxg'$ and $\bar{m}(\overline{g} \otimes \overline{g'}) = \overline{gxg'}$ which is easily checked to be well-defined. The right-most arrow is reduction of the two intermediate factors along $B_0 \to B_0/W_0$.

Now consider the element $fb \otimes b'f' \in L_n \otimes L_{n'}$, for the given $f \in L_n$ and $f' \in L_{n'}$. By assumption $\theta(fb \otimes b'f') = \overline{fb} \otimes \overline{b'f'}$ is non-zero, because $\overline{fb}, \overline{bf'} \neq 0$. Furthermore $\psi_n(\overline{fb})$ ends in B_0 and $\psi_{n'}(\overline{b'f'})$ begins in B_0, so $\psi_n(\overline{fb}) \otimes \psi_{n'}(\overline{b'f'})$ is in the domain of definition of the right-most arrow, which takes this element to $\psi_{n+n'+1}(\overline{fbxbf'})$. It remains to show that this element is nonzero. But $\psi_n(\overline{fb})$ does not end in W_0, and $\psi_{n'}(\overline{b'f'})$ does not begin in W_0; hence their images in $B \otimes \overline{B_0} \otimes \cdots \otimes \overline{B_0} \otimes B$ are nonzero, and their tensor product, equal to $\psi_{n+n'+1}(\overline{fbxbf'})$, is nonzero as well. $\square$

Proposition 3.2. *Let (K, ν) be a valued field containing F as a field of scalars. For any $m \geq 0$, $W = \{k \in K : \nu(k) \geq -m\}$ is a restricted subspace of K (where the decomposition $K = F \oplus K_0$ is arbitrary).*

Proof. We first claim that if $U \subseteq K$ is an F-vector subspace of finite codimension, then $\{\nu(u) : u \in U\}$ is unbounded from below. Indeed, choose any finite dimensional complement U', and notice that $\{\nu(u') : u' \in U'\}$ is bounded from below; so if $\nu(u)$ were bounded for $u \in U$, then $\nu(k)$ would be bounded over the set of $k \in K$.

Let $V \neq 0$ be a finite dimensional space. The space of elements y such that $Vy \subseteq K_0$ has finite codimension, so by the previous argument contains elements of arbitrarily small value, for which $Vy \not\subseteq W$. $\square$

Corollary 3.3. *Let (K, ν) be a valued field containing F as a field of scalars. For fixed $m \geq 0$, let $W_m = \{k \in K : \nu(k) \geq -m\}$. Then the ideal generated by xW_mx in $R = F[x] *_F K$ is prime.*

With the notation of Corollary 3.3, we now formulate the promised counterexample:

Example 3.4 (radical of a chain union which is a maximal ideal)**.** *Let F, K, R and the W_m be as above. By definition $\bigcup_{m\geq 0} W_m = K$. Let P_m be the (prime) ideal generated by xW_mx. Then $P_1 \subseteq P_2 \subseteq \cdots$ is a chain of prime ideals in R, and $\bigcup_{m\geq 0} RxW_mxR = RxKxR = (RxR)^2$. The radical RxR is thus maximal, as $R/RxR \cong K$.*

4. Matrix constructions

This section shows that $\mathcal{P}^{\uparrow}$ and $\mathcal{SP}^{\uparrow}$ are independent: the algebra in Example 4.1 satisfies $\mathcal{P}^{\uparrow}$ but not $\mathcal{SP}^{\uparrow}$, and the algebra in Example 4.5 satisfies $\mathcal{SP}^{\uparrow}$ but not $\mathcal{P}^{\uparrow}$.

4.1. **$\mathcal{P}^{\uparrow}$ does not imply $\mathcal{SP}^{\uparrow}$.** As mentioned in the introduction, Kaplansky conjectured that a semiprime ring all of whose prime quotients are von Neumann regular, is regular. Fisher and Snider [4] proved that this is the case if the ring satisfies $\mathcal{SP}^{\uparrow}$ (also see [5, Thm. 1.17]), and gave a counterexample which lacks this property, due to Kaplansky and Lanski [5, Example 1.19]. We repeat the example and exhibit, in this ring, an explicit ascending chain of semiprime ideals whose union is not semiprime.

Example 4.1 (Kaplansky-Lanski)**.** *(A ring whose prime ideals are maximal, but without $\mathcal{SP}^{\uparrow}$). Let R be the ring of sequences of 2-by-2 matrices which eventually have the form $\begin{pmatrix} \alpha & \beta_n \\ 0 & \alpha \end{pmatrix}$ in the nth place, clearly a semiprime ring.*

Let I_n be the set of sequences in R, which are zero from the nth place onward. Clearly $R/I_n \cong R$, so the ideals are semiprime. However $\bigcup I_n$ is composed of sequences of matrices which are eventually zero, and aRa is eventually zero for $a = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \dots\right)$; hence $R/\bigcup I_n$ is not semiprime. On the other hand by the argument in [4], every prime ideal of R is maximal, so there are no infinite chains of primes and $\mathcal{P}^{\uparrow}$ holds trivially.

4.2. **$\mathcal{SP}^{\uparrow}$ does not imply $\mathcal{P}^{\uparrow}$.** In the rest of this section we investigate $\mathcal{P}^{\uparrow}$ and $\mathcal{SP}^{\uparrow}$ for rings of the form $\hat{A} = \begin{pmatrix} A & M \\ M & A \end{pmatrix}$ where A is an integral domain and $M \lhd A$ is a nonzero ideal. We show that they always satisfy $\mathcal{SP}^{\uparrow}$, and give an example which does not have $\mathcal{P}^{\uparrow}$. Clearly $\hat{A}$ is a prime ring. Let us describe the ideals of this ring.

Proposition 4.2. (1) *The ideals of $\hat{A}$ have the form $\hat{I} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$, where for $1 \leq i, j \leq 2$, $I_{ij} \trianglelefteq A$ (not necessarily proper), $I_{ii'} \subseteq M$, and $MI_{ij} \subseteq I_{i'j} \cap I_{ij'}$ (where $1' = 2$ and $2' = 1$).*

(2) *The semiprime ideals of $\hat{A}$ are of the form*

$$\begin{pmatrix} I & M \cap I \\ M \cap I' & I' \end{pmatrix},$$

where I, I' are semiprime ideals of A, and $M \cap I' = M \cap I$.

Proof. (1) This is well known and easy.

(2) Write $A_{ij} = A$ if $i = j$ and $A_{ij} = M$ otherwise. We are given an ideal $\hat{I} \lhd \hat{A}$ which thus can be written as $\hat{I} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$, satisfying the conditions of (1). Clearly $\hat{I}$ is semiprime if and only if for every $a_{ij} \in A_{ij}$,

$$\left(\sum_{j,k} A_{jk} a_{ij} a_{k\ell} \subseteq I_{i\ell} \text{ for every } i, \ell \right) \text{ implies } \left(a_{i\ell} \in I_{i\ell} \text{ for every } i, \ell \right).$$

Assuming that this condition holds, fix i, j and choose $a_{k\ell} = 0$ for every $(k, \ell) \neq (i, j)$; then

$$(\diamondsuit) \text{ for } a_{ij} \in A_{ij},\ A_{ji} a_{ij}^2 \subseteq I_{ij} \text{ implies } a_{ij} \in I_{ij}.$$

On the other hand if Condition $(\diamondsuit)$ holds and $\sum_{j,k} A_{jk} a_{ij} a_{k\ell} \subseteq I_{i\ell}$ for every i, ℓ, then in particular $A_{ji} a_{ij}^2 \subseteq I_{ij}$, so each $a_{i\ell} \in I_{i\ell}$. We conclude that $\hat{I}$ is semiprime if and only if $(\diamondsuit)$ holds for every i, j.

We claim that $(\diamondsuit)$ is equivalent to I_{ii} being semiprime in A with

$$M \cap I_{11} \subseteq I_{ij}, \qquad \forall i \neq j.$$

Indeed, for $i = j$, condition $(\diamondsuit)$ requires that the I_{ii} are semiprime in A. Assuming this, the condition is "for $a_{ij} \in A_{ij}$, $M a_{ij}^2 \subseteq I_{ij}$ implies $a_{ij} \in I_{ij}$ for $i \neq j$." In light of the standing assumption that $a_{ij} \in A_{ij} = M$, we claim that this is equivalent to $M \cap I_{11} \subseteq I_{ij}$. Indeed, for every $b \in M$, $Mb^2 \subseteq I_{ij}$ iff $b \in I_{11}$ (Proof: If $b^2 M \subseteq I_{ij}$ then $b^4 \in (bM)^2 = b^2 M \cdot M \subseteq I_{ij} M \subseteq I_{11}$, so $b \in I_{11}$. On the other hand if $b \in M \cap I_{11}$ then $b^2 M \subseteq M I_{11} \subseteq I_{ij}$), so the condition becomes "for $b \in M$, $b \in I_{11}$ implies $b \in I_{ij}$ for $i \neq j$", as claimed.

We have shown that $\hat{I}$ is semiprime if and only if I_{11}, I_{22} are semiprime in A and $M \cap I_{11} \subseteq I_{12} \cap I_{21}$.

Now assume that I_{ii} are semiprime, and that $M \cap I_{11} \subseteq I_{12} \cap I_{21}$. Denote the idealizer of an ideal I by $(I : M) = \{x \in A : xM \subseteq I\}$, and notice that $M \cap (I : M) = M \cap I$ when I is semiprime. But $I_{12} M \subseteq I_{11}$, implying

$$I_{12} \subseteq M \cap (I_{11} : M) = M \cap I_{11} \subseteq I_{12},$$

so $I_{12} = M \cap I_{11}$ and likewise $I_{21} = M \cap I_{11}$. By symmetry $I_{12} = M \cap I_{22}$ as well, so $M \cap I_{11} = M \cap I_{22}$.

□

Proposition 4.3. *The ring $\hat{A}$ satisfies $\mathcal{SP}^{\uparrow}$.*

Proof. By Proposition 4.2 every chain of semiprime ideals $T_1 \subseteq T_2 \subseteq \cdots$ in $\hat{A}$ has the form $T_n = \begin{pmatrix} I_n & J_n \\ J_n & I'_n \end{pmatrix}$, I_n and I'_n are ascending chains of semiprime ideals of A, and $J_n = M \cap I_n = M \cap I'_n$. The union of this chain is $\begin{pmatrix} \bigcup I_n & L \\ L & \bigcup I'_n \end{pmatrix}$ where $L = M \cap \bigcup I_n = M \cap \bigcup I'_n$, which is semiprime. □

Using the description of the semiprime ideals, it is not difficult to obtain the following.

Proposition 4.4. (1) *The prime ideals of $\hat{A}$ are* $\begin{pmatrix} J & M \\ M & A \end{pmatrix}$ *and* $\begin{pmatrix} A & M \\ M & J \end{pmatrix}$ *for prime ideals $J \lhd A$ containing M, and* $I^0 = \begin{pmatrix} I & M \cap I \\ M \cap I & I \end{pmatrix}$ *for prime ideals $I \lhd A$ not containing M.*

(2) *The union-prime ideals of $\hat{A}$ are of the form* $\begin{pmatrix} M' & M \\ M & M' \end{pmatrix}$ *where $M' \lhd A$ is a prime ideal containing M, which can be written as a union of an ascending chain of primes not containing M.*

Proof. Notation as in Proposition 4.2(1), we are done by Proposition 4.2(2) unless some $I_{11} = A$, in which case we must have (1). If the chain of primes includes an ideal with A in one of the corners, then every higher term has the same form, and the union is determined by the union of the ideals in the other corner, which is prime since A is commutative. We thus assume that the chain has the form $I_1^0 \subseteq I_2^0 \subseteq \cdots$ where $I_1 \subseteq I_2 \subseteq \cdots$ are primes in A, not containing M. The union is clearly $\hat{I}^0$ where $\hat{I} = \bigcup I_n$ is prime, and $\hat{I}^0$ is not a prime iff $M \subseteq \hat{I}$. $\square$

Suppose A is a prime PI-ring, integral over its center C. In [3] it is shown that A satisfies the properties Lying Over and Going Up over C, which gives a correspondence of chains of primes between the two rings. The next example shows that the union of chains in not preserved.

Example 4.5 (A prime PI-ring, integral over its center, satisfying $\mathcal{SP}^{\uparrow}$ but not (uniquely-)$\mathcal{P}^{\uparrow}$)**.** *Let F be a field. Let* $\hat{A} = \begin{pmatrix} A & M \\ M & A \end{pmatrix}$, *where $A = F[\lambda_1, \dots]$ is the ring of polynomials in countably many variables $\lambda_1, \lambda_2, \dots$, and $M = \langle \lambda_1, \dots \rangle$. Clearly $\hat{A} \subset M_2(A)$ is integral over A. Choose $I_n = \langle \lambda_1, \dots, \lambda_n \rangle$. Then* $T_n = \begin{pmatrix} I_n & I_n \\ I_n & I_n \end{pmatrix} \lhd \hat{A}$ *form an ascending chain of primes by Proposition 4.4.(1), but their union* $\bigcup T_n = \begin{pmatrix} M & M \\ M & M \end{pmatrix}$ *is obviously not prime. Therefore $\hat{A}$ satisfies $\mathcal{SP}^{\uparrow}$ (Proposition 4.3) but not $\mathcal{P}^{\uparrow}$. Furthermore $\hat{A}/\bigcup T_n \cong A/M \times A/M$ which has two minimal ideals, so uniquely-$\mathcal{P}^{\uparrow}$ also fails.*

5. The $\mathcal{P}^{\uparrow}$-index

Let $\tilde{P} = \{P_\alpha\}$ be a chain of prime ideals in a ring R. The number of non-prime unions of subchains of $\tilde{P}$ is called the **index** of $\tilde{P}$ (either finite or infinite). The **$\mathcal{P}^{\uparrow}$-index** of R, denoted by $\mathcal{P}^{\uparrow}(R)$, is the supremum of the indices of all chains of primes in R.

Proposition 5.1. *For any ring R, $\mathcal{P}^{\uparrow}(R) = \sup \mathcal{P}^{\uparrow}(R/P)$ where $P \lhd R$ ranges over the prime ideals.*

Proof. By definition $\mathcal{P}^{\uparrow}(R/P)$ is the supremum of indices of chains of primes containing P. Therefore, the supremum on the right-hand side is the supremum of indices of chains of primes containing some prime P, but any chain contains its own intersection, so this supremum is by definition $\mathcal{P}^{\uparrow}(R)$. $\square$

Remark 5.2. *If $n = \mathcal{P}^{\uparrow}(R)$ then R has a chain of n union-primes. It is not clear if the converse holds. For example if $\{P_\lambda\}$ and $\{P'_\lambda\}$ are ascending chains of primes such that $\bigcup P_\lambda \subset \bigcup P'_\lambda$ are not primes, does it follow that there is a chain of primes with at least two non-prime unions?*

We claim that:

Proposition 5.3. *For any ring R,*

$$\mathcal{P}^{\uparrow}(R) = \begin{cases} 0 & \text{if } R \text{ has the property } \mathcal{P}^{\uparrow} \\ \sup_I \mathcal{P}^{\uparrow}(R/I) + 1 & \text{otherwise} \end{cases}$$

where the supremum is taken over the union-prime ideals of R (when they exist).

Proof. Indeed, $\mathcal{P}^{\uparrow}(R) = 0$ if and only if there are no union-primes (which are non-prime by definition), if and only if R satisfies $\mathcal{P}^{\uparrow}$. Now assume R does not satisfy $\mathcal{P}^{\uparrow}$. Consider the set $\{\mathcal{P}^{\uparrow}(R/I)\}$ ranging over the union-primes I. If this set is unbounded, then clearly $\mathcal{P}^{\uparrow}(R) = \infty$. Otherwise, take a union-prime I such that $n = \mathcal{P}^{\uparrow}(R/I)$ is maximal among the $\mathcal{P}^{\uparrow}$-indices of the quotients. If $J_1/I \subset \cdots \subset J_n/I$ are union-primes in a chain of primes in R/I, then $I \subset J_1 \subset \cdots \subset J_n$ are union-primes in a chain in R. On the other hand if $J_0 \subset J_1 \subset \cdots \subset J_n$ are union-primes in a chain in R, then $\mathcal{P}^{\uparrow}(R/J_0) \geq n$. $\square$

For example, $\mathcal{P}^{\uparrow}(R) = 1$ if and only if the union of an ascending chain of primes starting from a union-prime ideal is necessarily prime.

6. The property $\mathcal{P}^{\uparrow}$ in PI-rings

In this section we show that for PI-rings, the $\mathcal{P}^{\uparrow}$-index is bounded by the PI-class.

Proposition 6.1. *Any Azumaya algebra satisfies $\mathcal{P}^{\uparrow}$ (and $\mathcal{SP}^{\uparrow}$).*

Proof. Let A be an Azumaya algebra over a commutative ring C. There is a 1:1 correspondence between ideals of A and the ideals of C, preserving inclusion, primality and semiprimality. The claim follows since the center satisfies $\mathcal{P}^{\uparrow}$ (and $\mathcal{SP}^{\uparrow}$). $\square$

Recall that by Posner's theorem ([10]), a prime PI-ring R is *representable*, namely embeddable in a matrix algebra $\mathrm{M}_n(C)$ over a commutative ring C. The minimal such n is the PI-class of R, denoted $\mathrm{PI}(R)$.

Although PI-rings do not necessarily satisfy the property $\mathcal{P}^{\uparrow}$, we show that the PI-class bounds the extent in which $\mathcal{P}^{\uparrow}$ may fail.

We are now ready for our main positive result about PI-rings.

Theorem 6.2. *Let R be a (prime) PI-ring. Then $\mathcal{P}^{\uparrow}(R) < \mathrm{PI}(R)$.*

Proof. Let R be a prime PI-ring of PI-class n. If the PI-class is 1 then R is commutative, and has $\mathcal{P}^{\uparrow}(R) = 0$. We continue by induction on n. Let

$$0 = P_0 \subset P_1 \subset \cdots$$

be an ascending chain of primes, and assume that $\bigcup P_n$ is not a prime ideal. Let $Q \supset \bigcup P_n$ be a prime ideal. We want to prove that the PI-class of R/Q is smaller than that of R.

Assume otherwise. Let g_n be a central polynomial for $n \times n$ matrices (see [10, p. 26]). Since $\mathrm{PI}(R/Q) = n$, there is a value $\gamma \neq 0$ of g_n in the center of R, which is not in Q. Since the center is a domain we can consider the localization $A[\gamma^{-1}]$ (see [11, Section 2.12]), which is Azumaya by Rowen's version of the Artin-Procesi

Theorem [10, Theorem 1.8.48], since 1 is a value of g_n on this algebra. But then the union of

$$0 \subset P_1[\gamma^{-1}] \subset P_2[\gamma^{-1}] \subset \cdots$$

is prime by Proposition 6.1, so $\bigcup P_n$ is prime as well, contrary to assumption. □

We now show the bound is tight. Notice that the ring constructed in Example 4.5 has PI-class 2 and is not $\mathcal{P}^{\uparrow}$(and thus has $\mathcal{P}^{\uparrow}(R) = 1$). Let us generalize this.

Example 6.3 (An algebra of PI-class n which has $\mathcal{P}^{\uparrow}$-index $n-1$)**.** *Let* $A_{(n)} = F\left[\lambda_i^{(j)} : 1 \leq j < n,\ i = 1, 2, \dots\right]$. *Let* $M_n = 0$ *and, for* $j = n-1, n-2, \dots, 1$, *take* $M_j = M_{j+1} + \left\langle \lambda_1^{(j)}, \lambda_2^{(j)}, \cdots \right\rangle$, *so that* $0 = M_n \subset M_{n-1} \subset \cdots \subset M_1 \lhd A_{(n)}$. *Let* e_{ij} *denote the matrix units of the matrix algebra over* $A_{(n)}$. *Let* $J_{(n)} = \sum_{i,j} e_{ij} M_{\max(i,j)-1}$, $S_{(n)} = \sum e_{ii} A_{(n)}$. *Let*

$$R_{(n)} = J_{(n)} + S_{(n)} = \begin{pmatrix} A_{(n)} & M_1 & M_2 & \cdots & M_{n-1} \\ M_1 & A_{(n)} & M_2 & \cdots & M_{n-1} \\ M_2 & M_2 & A_{(n)} & \cdots & M_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-1} & M_{n-1} & \cdots & A_{(n)} \end{pmatrix}.$$

Clearly $S_{(n)} J_{(n)}$, $J_{(n)} S_{(n)} \subseteq J_{(n)}$, *and* $S_{(n)} S_{(n)} = S_{(n)}$. *Moreover* $M_k M_\ell \subseteq M_{\max\{k,\ell\}}$ *for every* k, ℓ, *so that* $J_{(n)} J_{(n)} \subseteq J_{(n)}$. *It follows that* $R_{(n)} = J_{(n)} + S_{(n)}$ *is a ring. The ring of central fractions of* $R_{(n)}$ *is the simple ring* $\mathrm{M}_n(\mathrm{q}(A_{(n)}))$, *so* $R_{(n)}$ *is prime, of PI-class* n.

When $n = 2$ *we obtain the ring of Example 4.5, so* $\mathcal{P}^{\uparrow}(R_{(2)}) = 1$. *For arbitrary* n, *consider the chain* $I_1 \subset I_2 \subset \cdots$ *of ideals of* A *defined by* $I_i = \left\langle \lambda_1^{(n-1)}, \dots, \lambda_i^{(n-1)} \right\rangle$; *thus* $\bigcup I_i = M_{n-1}$. *Let* $\tilde{I}_i = \mathrm{M}_n(I_i)$. *Each ideal* $\tilde{I}_i$ *is prime (again by central fractions), and their union is the set of matrices over* M_{n-1}. *The quotient ring is therefore* $R_{(n)}/\bigcup_i \tilde{I}_i \cong R_{(n-1)} \times A_{(n-1)}$, *which is not prime, and* $\mathcal{P}^{\uparrow}(R_{(n-1)}) = n-2$ *by induction. We conclude that* $\mathcal{P}^{\uparrow}(R_{(n)}) = n - 1$.

Remark 6.4.
1. *Although PI-rings do not necessarily satisfy* $\mathcal{P}^{\uparrow}$*(see Example 4.5), affine PI-rings over commutative Noetherian rings do satisfy ACC on semiprime ideals, and in particular are* $\mathcal{SP}^{\uparrow}$ *and* $\mathcal{P}^{\uparrow}$ *(by Schelter's theorem,* [10, Thm 4.4.16]*).*
2. *PI-rings of finite Gel'fand-Kirillov dimension satisfy ACC on primes, since a prime PI-ring is Goldie, and then every prime ideal contains a regular element which reduces the dimension.*
3. *On the other hand, we have examples of a (non-affine) locally nilpotent monomial algebra, which does not satisfy* $\mathcal{P}^{\uparrow}$, *and of affine algebras of* $\mathrm{GKdim} = 2$ *which do not satisfy* $\mathcal{P}^{\uparrow}$ *(details will appear elsewhere.) In both cases the* $\mathcal{P}^{\uparrow}$*-index is uncountable.*
4. *Graded affine algebras of quadratic growth and graded affine domains of cubic growth have finite Krull dimension and so satisfy* $\mathcal{P}^{\uparrow}$, [2, 13].

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