Medial solutions to QYBE
Part I

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Algebra

$$(X, F)$$

$X \neq \emptyset$
F - a set of operations $f\colon X^n \to X$

$(X, \circ)$ - a groupoid: an algebra with one binary operation $\circ\colon X^2 \to X$

For each $s \in X$

$$L_s\colon X \to X;\; x \mapsto s \circ x$$

the left translation with respect to the operation $\circ$

$$R_s\colon X \to X;\; x \mapsto x \circ s$$

the right translation with respect to the operation $\circ$

(X, r) - Quadratic set

$r \colon X^2 \to X^2$ a bijection

(X, r) - Quadratic set

$$r\colon X^2 \to X^2 \text{ a bijection}$$

$$r(x,y) = (x \circ y, x \bullet y) = (L_x(y), R_y(x))$$

L_x - the left translation with respect to the operation $\circ$

R_y - the right translation with respect to the operation $\bullet$

(X, r) - Set-theoretical solution of Yang-Baxter equation

Braid relation:

$$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r)$$

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Braid identities in $(X, \circ, \bullet)$

$$\begin{aligned} x \circ (y \circ z) &= (x \circ y) \circ ((x \bullet y) \circ z) \\ (x \circ y) \bullet ((x \bullet y) \circ z) &= (x \bullet (y \circ z)) \circ (y \bullet z) \\ x \bullet (y \bullet z) &= (x \bullet (y \circ x)) \bullet (y \bullet z) \end{aligned}$$

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$$x \circ (x \backslash y) = y, \qquad x \backslash (x \circ y) = y$$
$$(y/x) \bullet x = y, \qquad (y \bullet x)/x = y$$

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- Left quasi-group $(X, \circ) \rightleftharpoons (X, \circ, \backslash)$
Right quasi-group $(X, \bullet) \rightleftharpoons (X, \bullet, /)$

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$$r^2(x, y) = (x, y)$$

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Involutive identities in $(X, \circ, \bullet)$

$$(x \circ y) \circ (x \bullet y) = x$$
$$(x \circ y) \bullet (x \bullet y) = y.$$

(X, r) - Square free solution

$$r(x, x) = (x, x)$$

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$$r(x, x) = (x, x)$$

$(X, \circ, \bullet)$ is *idempotent*, i.e. for every $x \in X$:

$$x \circ x = x$$

$$x \bullet x = x$$

Biracks

Each non-degenerate set-theoretical solution of Yang-Baxter equation (X, r) yields an algebra $(X, \circ, \bullet)$ such that

- $(X, \circ, \backslash)$ is a left quasi-group
- $(X, \bullet, \backslash)$ is a right quasi-group
- $(X, \circ, \bullet)$ satisfies braid identities

$(X, \circ, \bullet)$ - **birack**

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$(X, \circ, \bullet)$ - (involutive) **birack**

Theorem (Rump; Gateva-Ivanova; Dehornoy)

If $(X, \circ, \bullet)$ is an (involutive) birack then (X, r) is a non-degenerate (involutive) solution of Yang-Baxter equation with $r(x, y) = (x \circ y, x \bullet y)$

Involutive birack

For an involutive birack $(X, \circ, \bullet)$:

The right quasi-group $(X, \bullet, /)$ is completely determined by the left quasi-group $(X, \circ, \backslash)$

$$x \bullet y = L^{-1}_{x \circ y}(x) = (x \circ y)\backslash x$$

Right cyclic left quasi-group

Right cyclic law in left quasi-group $(X, *, \backslash_*)$

$$(x * y) * (x * z) = (y * x) * (y * z)$$

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Theorem (Rump; Gateva-Ivanova; Dehornoy)

If $(X, \circ, \bullet)$ is a non-degenerate involutive birack then $(X, \backslash, \circ)$ is a right cyclic left quasi-group

Right cyclic left quasi-group

Right cyclic law in left quasi-group $(X, *, \backslash_*)$

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Theorem (Rump; Gateva-Ivanova; Dehornoy)

If $(X, \circ, \bullet)$ *is a non-degenerate involutive birack then* $(X, \backslash, \circ)$ *is a right cyclic left quasi-group*

If $(X, *, \backslash_*)$ *is a right cyclic left-quasigroup then* $(X, \circ, \bullet)$ *is an involutive birack with* $x \circ y = x\backslash_* y$ *and* $x \bullet y = x\backslash_* y * x$

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Remark

To find all involutive non-degenerate solutions of Yang-Baxter equation is equivalent to construct all right cyclic left-quasigroups.

Examples

Example

$(B, \cdot, +)$ - a left brace
$(B, *, \backslash_*)$ with $x * y = x^{-1}(x + y)$ and $x \backslash_* y = xy - x$ is a right cyclic left-quasigroup

Example

$(A, +)$ - an abelian group
f - automorphism of $(A, +)$ such that $(id - f)^2$ is nilpotent of degree 2
$c \in ker(id - f)$
$(A, *, \backslash_*)$ with

$$x * y = f^{-1}(y - (id - f)(x) - c) \quad \text{and} \quad x \backslash_* y = (id - f)(x) + f(y) + c$$

is a right cyclic left-quasigroup

Left-distributivity

Definition

$(X, *)$ is **left-distributive** if for every $x, y, z \in X$,

$$x * (y * z) = (x * y) * (x * z)$$

All left translations $\ell_x \colon X \to X; a \mapsto x * a$, for every $x \in X$, are endomorphisms of $(X, *)$, i.e. for $y, z \in X$,

$$\ell_x(y * z) = \ell_x(y) * \ell_x(z).$$

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Definition

A left quasi-group $(X, *, \backslash_*)$ is left distributive if $(X, *)$ is left-distributive. A left-distributive left quasi-group is called a **rack**.

Mutually orthogonal quasi-groups

$(X, \circ, \bullet)$ - an involutive left distributive birack

- $R_y(x) = x \bullet y = y \backslash x = L_y^{-1}(x)$ and
  $R_y^{-1}(x) = x/y = y \circ x = L_y(x)$
- the left quasi-group $(X, \circ, \backslash)$ and the right quasi-group $(X, \bullet, /)$ are **mutually orthogonal**, i.e. for every $a, b \in X$, the pair of equations

$$a = x \circ y \ \text{ and } \ b = x \bullet y$$

  has a unique solution

2-reductivity

Definition

A groupoid $(X, *)$ is 2-*reductive* if for every $x, y, z \in X$,

$$(x * y) * z = y * z.$$

Example

p - a prime number
The groupoid $(Z_{p^2}, *)$ with

$$x * y = px + (1 - p)y$$

is 2-reductive

Multipermutation solution

Theorem (Gateva-Ivanova)

If $(X, \circ, \bullet)$ is an involutive birack satisfying the condition:

$$(*) \quad \text{for every } x \in X \text{ there exists some } a \in X \text{ with } a \circ x = x,$$

then $(X, \circ, \bullet)$ is a multipermutation solution level less or equal to 2 if and only if $(X, \circ)$ is 2-reductive.

In particular, this is true for idempotent biracks (square free solutions).

2-reductivity

Lemma

$(X, \circ, \backslash)$ - a rack
$(X, \circ)$ is 2-reductive if and only if the group

$$\mathrm{LMlt}(X, \circ) = \langle L_x \mid x \in X \rangle$$

is abelian

Corollary

$(X, \circ, \bullet)$ - an involutive birack
$(X, \circ)$ is left distributive if and only if it is 2-reductive

Involutive left distributive biracks

Theorem

$(X, \circ, \bullet)$ - an involutive left distributive birack
The following are equivalent:

- $(X, \circ, \bullet)$ *is a multipermutation solution level 2*
- $(X, \circ)$ *is 2-reductive*
- *the group* $\mathrm{LMlt}(X, \circ)$ *is abelian*

Sum of trivial affine mesh

Trivial affine mesh over $I \neq \emptyset$

$$\mathcal{A} = ((A_i)_{i\in I};\ (c_{i,j})_{i,j\in I})$$

A_i - abelian group for each $i \in I$
$c_{i,j} \in A_j$ - constants such that for every $j \in I$

$$A_j = \langle \{c_{i,j} \mid i \in I\} \rangle$$

Definition

The *sum of a trivial affine mesh* $\mathcal{A}$ over a set I: $(\bigcup_{i\in I} A_i, \circ, \backslash)$

$$a \circ b = b + c_{i,j}$$
$$a \backslash b = b - c_{i,j}$$

for every $a \in A_i$ and $b \in A_j$

Medial racks

Theorem

An algebra is a 2-reductive rack if and only if it is the sum of trivial affine mesh.

The orbits of the rack coincide with the groups of the mesh.

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$(X, *)$ - 2-reductive

$(X, *)$ is left distributive if and only if $(X, *)$ is medial, i.e. for every $x, y, z, t \in X$,

$$(x * y) * (z * t) = (x * z) * (y * t).$$