Medial solutions to QYBE

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Noncommutative and non-associative structures, braces and applications

Racks and quandles

A binary algebra $(Q, *)$ is called a **rack** if it is:

- a **left quasigroup**: the equation $x * u = y$ has a unique solution $u \in Q$ for every $x, y \in Q$,
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A **quandle** is an **idempotent** rack: $x * x = x$ for each $x \in Q$.

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If $*$ is idempotent, then $\backslash$ is idempotent, too.

Two important groups

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Both groups are normal subgroups of $\mathrm{Aut}(Q)$.

Two important groups

Proposition (Joyce; HSV)

Let Q be a rack. Then

1. $\mathrm{Dis}(Q) = \{L_{a_1}^{k_1} \dots L_{a_n}^{k_n} : \ a_1, \dots, a_n \in Q \text{ and } \sum_{i=1}^{n} k_i = 0\}$;
2. *if Q is a quandle, the natural actions of $\mathrm{LMlt}(Q)$ and $\mathrm{Dis}(Q)$ on Q have the same orbits;*
3. $\mathrm{Dis}(Q)$ *is abelian if and only if Q is* **medial**.

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A rack Q is called **medial** if, for every $x, y, u, v \in Q$,

$$(x * y) * (u * v) = (x * u) * (y * v),$$
$$(x \backslash y) \backslash (u \backslash v) = (x \backslash u) \backslash (y \backslash v),$$
$$(x \backslash y) * (u \backslash v) = (x * u) \backslash (y * v).$$

Orbit decomposition for quandles

Orbits of a quandle Q are the orbits of transitivity of the groups $\mathrm{LMlt}(Q)$ and $\mathrm{Dis}(Q)$ denoted

$$Qe = \{\alpha(e) \mid \alpha \in \mathrm{LMlt}(Q)\} = \{\alpha(e) \mid \alpha \in \mathrm{Dis}(Q)\},$$

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For medial quandle, orbits are **Alexander quandles**.

$(A, +)$ - an abelian group, $f \in \mathrm{Aut}(A)$. A quandle $(A, *, \backslash)$, where

$$x * y = x - f(x) + f(y) = (1 - f)(x) + f(y),$$
$$x \backslash y = (1 - f^{-1})(x) + f^{-1}(y),$$

is called an **Alexander quandle** (or **affine quandle**).

Reductivity

A left quasi-group $(Q, *, \backslash)$ is ***m*-reductive**, if it satisfies for $x, y_1, y_2, y_3, \ldots, y_m \in Q$

$$(((x * y_1) * y_2) * \ldots) * y_m = (((y_1 * y_2) * y_3) * \ldots) * y_m.$$

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For medial quandles m-reductivity is equivalent to the following:

$$(((x * \underbrace{y) * y) * \dots) * y}_{m-\text{times}} = y.$$

Reductive racks

Theorem

Let Q be a rack. Then the following properties are equivalent:

1. *Q is m-reductive,*
2. $\mathrm{LMlt}(Q)$ *is* $(m-1)$*-nilpotent,*
3. *Q is a multipermutation rack of class m.*

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For a rack Q the relation λ defined by

$$a \;\lambda\; b \;\text{ iff }\; \forall (x \in Q)\; a * x = b * x \;\text{ iff }\; \forall (x \in Q)\; a \backslash x = b \backslash x,$$

is a **congruence**.

Multipermutation solution

Non-degenerate involutive solutions of QYBE $\Longleftrightarrow$ involutive biracks $\Longleftrightarrow$ right cyclic left-quasigroups

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Theorem (Gateva-Ivanova, 2017)

Let $(Q, \circ, \bullet)$ be an involutive birack satisfying

$$\text{for every } x \in Q \text{ there exists some } a \in Q \text{ with } a \circ x = x.$$

TFAE

- *Q is a multipermutation solution level equal to m*
- *$(Q, \circ)$ is m-reductive.*

Reductivity in (medial) quandles

n	1	2	3	4	5	6	7	8	9	10
all	1	1	3	7	22	73	298	1581	11079	
medial	1	1	3	6	18	58	251	1410	10311	98577
non-reductive	0	0	1	1	3	1	5	3	10	3

n	11	12	13	14	15
medial	1246488	20837439	466087635		563753074951
non-reductive	9	8	11	5	24

Table: The number of quandles of size n, up to isomorphism.

Subdirectly irreducible medial quandles

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For reductive case the monolith is $\theta := \pi \cap \lambda$.

Relation θ

Theorem (Cedó, Jespers, Okniński, 2009)

Let $(Q, \circ, \bullet)$ *be an idempotent, involutive birack with the group* $\mathrm{LMlt}(Q)$ *being abelian. Then the solution* (Q, r) *is strongly retractable.*

THANK YOU FOR YOUR ATTENTION!