Growth, relations and prime spectra of monomial algebras

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How algebras grow?

R- finitely generated associative algebra over a field F.
V- fin. dim. generating subspace, $1 \in V$.

Definition

The growth *of R is the asymptotic behavior of the sequence* $\dim_F V^n$*.*

Remark

- *The growth is indpt. of choice of V (up to: $f \sim g$ iff $f(n) \leq Cg(Dn) \leq C'f(D'n)$)*
- *Polynomial, intermediate, exponential*
- *If polynomially bounded: $GKdim(R) = \limsup_{n\to\infty} \log_n(\dim_F V^n)$*
- *If R is commutative then it grows $\sim n^d$ where $d = Krull(R) = GKdim(R)$*

Growth of algebras: Importance and applications

- GK-dimension = dimension of noncommutative projective schemes
- GK-dimension plays important role in theory of D-modules, holonomicity (Bernstein's inequality...)
- $GKdim(R) \in \{0\} \cup \{1\} \cup [2, \infty]$ (Bergman's gap)
- Allows to define 'noncommutative transcendence degree' = invariant for division algebras (even with exponential growth)
- Groups of intermediate growth (e.g. Grigorchuk's group) give rise to algebras of intermediate growth
- Algebras of subexponential growth are amenable
- Much more in NC-geometry, combinatorial algebra, geometric group theory...

Realizing growth functions

A natural question arises: which functions describe the growth rate of an algebra? (For groups, very little is known)

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Theorem (Bartholdi-Smoktunowicz, '14)

If f satisfies the above assumptions then there is an algebra R with growth function:

$$f(n) \preceq \gamma_R(n) \preceq n^2 f(n)$$

In particular, if $\exists C$ such that $f(Cn) \geq nf(n)$ (any sufficiently regular function more rapid than $n^{\log n}$) then $\gamma_R \sim f$.

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However, they do not treat algebraic properties of the realizing algebras; they pose the question of whether their resulting algebras are (or can be made) prime.

The Bartholdi-Smoktunowicz construction

Consider free algebra $F\langle x_1, \ldots, x_d \rangle$. Inductively define for $n \geq 0$:

- $W(1) = \{x_1, \ldots, x_d\}$;
- $C(2^n) \subseteq W(2^n)$ arbitrary;
- $W(2^{n+1}) = C(2^n)W(2^n)$.

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Mod out the free algebra by all monomials which are not subwords of monomials from $\bigcup_{n\geq 0} W(2^n)$. We get an algebra spanned by all finite subwords of words from the infinite set:

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If $|C(2^n)| = f(2^{n+1})/f(2^n)$ then the factor algebra has growth $f(n) \preceq \gamma_R(n) \preceq n^2 f(n)$.

Lemma (G., 2016)

If every $w \in W(2^n)$ is the suffix of some $v \in C(2^N)$ with $N \geq n$, then the factor algebra is prime.

Alahmadi-Alsulami-Jain-Zelmanov conjecture

Recall that a primitive algebra is an algebra admitting a faithful simple module. Every primitive algebra is prime.

Conjecture (Alahmadi-Alsulami-Jain-Zelmanov, 2017)

If $f : \mathbb{N} \to \mathbb{N}$ is sufficiently rapid and realizable as growth function of a finitely generated algebra, then it is realizable as the growth function of a finitely generated primitive *algebra.*

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The largest known source for growth rate functions arises from Bartholdi-Smoktunowicz.

Theorem (G., 2016)

If f satisfies the conditions of the Bartholdi-Smoktunowicz construction (submultiplicative, $\exists C : f(Cn) \geq nf(n)$) then there exists a primitive algebra with growth function $\sim f$.

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Note: under additional mild rapidness condition we are able to realize with simple algebras (convolution algebras of appropriate étale groupoids).

Primitive algebras

Proof idea:

- Construct an inverse systems of monomial algebras, each of which arises from the Bartholdi-Smoktunowicz construction:

$$\cdots \to R_2 \to R_1$$

- The intersection of the defining ideals defines a 'limit' algebra R_∞ whose Jacobson radical we can vanish (carefully defining the inverse system - each finite step is *not* primitive);
- Prove the resulting algebra is prime (using the lemma);
- Deduce primitivity by Okniński's trichotomy for monomial algebras;
- Achieve precise control on growth of the limit algebra by careful analysis and 'sparse' enough choice of defining ideals along the inverse system.

Bergman's question

Recall that for finitely generated commtative algebras, $Krull(R) = GKdim(R)$. For PI-algebras, noetherian algebras (and others) we have: $cl.Krull(R) \leq GKdim(R)$ (cl.Krull = classical Krull dimension, maximum length of chain of prime ideals).

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Method: Affinization (embedding countable dim. alg. as 'corner' of f.g.)

- Uses Zorn's lemma - non-constructive
- No concrete/computable example
- No control on precise growth (only known $GKdim = 2$)

A concrete, tame example

We recall the following positive partial answer to Bergman's question:

Theorem (G.-Leroy-Smoktunowicz-Ziembowski, 2015)

If $R = \bigoplus_{i \in \mathbb{Z}} R_i$ is generated in degrees $-1, 0, 1$ and has quadratic growth ($\dim_F V^n \sim C \cdot n^2$) then $cl.Krull(R) \leq 2C + 4$ so no infinite chains of primes can occur.

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On the other hand:

Theorem (G., 2018)

Suppose $\omega(n) \to \infty$ *arbitrarily slowly (non-decreasing) and* $N > 1$ *given. Then there exists a monomial algebra (hence graded, gen. in deg. 1)* R *such that:*

- $n^2 \preceq \gamma_R(n) \preceq n^2\omega(n)$;
- $cl.Krull(R) > N$.

Algebras with prescribed relations

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(Motivation: Lenagan-Smoktunowicz example of nil algebra with finite GK-dim.)

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Answer (Smoktunowicz, 2013; Bartholdi-Smoktunowicz, 2014): YES!

Strengthened analogy for monomial relations

Can we construct monomial algebras with prescribed growth rate, satisfying prescribed monomial relations?

Theorem (G., 2018)

Suppose we have a set of monomial relations in the free algebra with r_n relations in degree n, and a submultiplicative function $f : \mathbb{N} \to \mathbb{N}$ such that:

- $r_{n+m} \leq r_n r_m$;
- $r_n = 0$ *for* $n < N$ *(for some large enough* N*);*
- $r_n \prec f(n)/n$

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- 'subexponentially many' relations $\implies$ algebra with intermediate growth (cf. Drensky's question)
- 'polynomially many' $\implies$ finite GK-dim (cf. Zelmanov's question)

Concluding questions

Question

Suppose R is graded, fin. gen. in deg. 1 with quadratic growth. Is it possible that $cl.Krull(R) > 2$?

Question

Is there a f.g. nil algebra of quadratic growth? Of $GKdim = 2$?

Thank You!

Questions?