Finite quotients of groups of I-type or Quantum Yang-Baxter groups

Fabienne Chouraqui

University of Haifa, Campus Oranim

Finite quotients of groups of I-type or Quantum Yang-Baxter groups

Joint work with Eddy Godelle, Caen

Finite quotients of groups of I-type 2014

Properties of a solution (X, r)

Let $X = \{x_1, ..., x_n\}$ and let r be defined in the following way: $r(i,j) = (\sigma_i(j), \gamma_j(i))$, where $\sigma_i, \gamma_i : X \to X$.

Proposition [P.Etingof, T.Schedler, A.Soloviev - 1999]

- (X, r) is non-degenerate $\Leftrightarrow$ σ_i and γ_i are bijective, $1 \leq i \leq n$.
- (X, r) is involutive $\Leftrightarrow$ $r^2 = Id_{X \times X}$.
- (X, r) is braided $\Leftrightarrow$ $(Id \times r)(r \times Id)(Id \times r) = (r \times Id)(Id \times r)(r \times Id)$

Multipermutations solutions of level $m \geq 1$

A *retract relation* $\equiv$ on X is defined by:

Multipermutations solutions of level $m \geq 1$

A *retract relation* $\equiv$ on X is defined by:

$x_i \equiv x_j$ if and only if $\sigma_i = \sigma_j$.

Multipermutations solutions of level $m \geq 1$

A *retract relation* $\equiv$ on X is defined by:

$x_i \equiv x_j$ if and only if $\sigma_i = \sigma_j$.

(X, r) is a*multipermutation solution of level m or retractable* if:

Multipermutations solutions of level $m \geq 1$

A *retract relation* $\equiv$ on X is defined by:

$x_i \equiv x_j$ if and only if $\sigma_i = \sigma_j$.

(X, r) is a*multipermutation solution of level m or retractable* if:

There exits $m \geq 1$ such that $\mathrm{Ret}^m(G)$ is a cyclic group and m is the smallest such integer, where $\mathrm{Ret}^{k+1}(G) = \mathrm{Ret}^1(\mathrm{Ret}^k(G))$.

The QYBE group: the structure group of (X, r)

Assumption: (X, r) is a non-degenerate, involutive and braided solution.

The QYBE group: the structure group of (X, r)

Assumption: (X, r) is a non-degenerate, involutive and braided solution.

The structure group G of (X, r) [Etingof, Schedler, Soloviev]

- The generators: $X = \{x_1, x_2, .., x_n\}$.

The QYBE group: the structure group of (X, r)

Assumption: (X, r) is a non-degenerate, involutive and braided solution.

The structure group G of (X, r) [Etingof, Schedler, Soloviev]

- The generators: $X = \{x_1, x_2, .., x_n\}$.
- The defining relations: $x_i x_j = x_k x_l$ whenever $r(i, j) = (k, l)$

The QYBE group: the structure group of (X, r)

Assumption: (X, r) is a non-degenerate, involutive and braided solution.

The structure group G of (X, r) [Etingof, Schedler, Soloviev]

- The generators: $X = \{x_1, x_2, .., x_n\}$.
- The defining relations: $x_i x_j = x_k x_l$ whenever $r(i, j) = (k, l)$

There are exactly $\frac{n(n-1)}{2}$ defining relations.

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define r

$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1, 2, 3, 4)$
$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1, 4, 3, 2)$

(X, r) is a non-degenerate, involutive and braided solution.
(X, r) is a multipermutation of level 2.

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define r

$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1, 2, 3, 4)$
$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1, 4, 3, 2)$

(X, r) is a non-degenerate, involutive and braided solution.
(X, r) is a multipermutation of level 2.

The defining relations in G and in M

$$\begin{array}{cc} x_1^2 = x_2^2 & x_3^2 = x_4^2 \\ x_1x_2 = x_3x_4 & x_1x_3 = x_4x_2 \\ x_2x_4 = x_3x_1 & x_2x_1 = x_4x_3 \end{array}$$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define r

$$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1,2,3,4)$$
$$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1,4,3,2)$$

(X, r) is a non-degenerate, involutive and braided solution.
(X, r) is a multipermutation of level 2.

The defining relations in G and in M

$$\begin{array}{llll} x_1^2 = x_2^2 & x_3^2 = x_4^2 & (x_1x_4 = x_1x_4 & x_2x_3 = x_2x_3) \\ x_1x_2 = x_3x_4 & x_1x_3 = x_4x_2 & & \\ x_2x_4 = x_3x_1 & x_2x_1 = x_4x_3 & (x_3x_2 = x_3x_2 & x_4x_1 = x_4x_1) \end{array}$$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define r

$$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1, 2, 3, 4)$$
$$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1, 4, 3, 2)$$

(X, r) is a non-degenerate, involutive and braided solution.
(X, r) is a multipermutation of level 2.

The defining relations in G and in M

$$\begin{array}{llll} x_1^2 = x_2^2 & x_3^2 = x_4^2 & (x_1x_4 = x_1x_4 & x_2x_3 = x_2x_3) \\ x_1x_2 = x_3x_4 & x_1x_3 = x_4x_2 & & \\ x_2x_4 = x_3x_1 & x_2x_1 = x_4x_3 & (x_3x_2 = x_3x_2 & x_4x_1 = x_4x_1) \end{array}$$

There are $\frac{n(n-1)}{2}$ relations (and n trivial relations)

The correspondence between QYBE groups and Garside groups

Theorem (F.C. 2009)

Let (X, r) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then G is Garside.

The correspondence between QYBE groups and Garside groups

Theorem (F.C. 2009)

Let (X, r) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then G is Garside.

Assume that $\mathrm{Mon}\langle X \mid R\rangle$ is a **Garside monoid** such that:
- the cardinality of R is $n(n-1)/2$
- each side of a relation in R has length 2.
- if the word $x_i x_j$ appears in R, then it appears only once.

The correspondence between QYBE groups and Garside groups

Theorem (F.C. 2009)

Let (X, r) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then G is Garside.

Assume that $\mathrm{Mon}\langle X \mid R\rangle$ is a **Garside monoid** such that:
- the cardinality of R is $n(n-1)/2$
- each side of a relation in R has length 2.
- if the word $x_i x_j$ appears in R, then it appears only once.

Then $G = \mathrm{Gp}\langle X \mid R\rangle$ is the structure group of a non-degenerate, involutive and braided solution (X, r), with $\mid X \mid = n$.

What are the advantages of being a Garside group?

What are the advantages of being a Garside group?

The advantages of being Garside

If the group G is Garside, then

- G is torsion-free [P.Dehornoy 1998]

What are the advantages of being a Garside group?

The advantages of being Garside

If the group G is Garside, then

- G is torsion-free [P.Dehornoy 1998]
- G is bi-automatic [P.Dehornoy 2002]

What are the advantages of being a Garside group?

The advantages of being Garside

If the group G is Garside, then

- G is torsion-free [P.Dehornoy 1998]
- G is bi-automatic [P.Dehornoy 2002]
- G has word and conjugacy problem solvable

What are the advantages of being a Garside group?

The advantages of being Garside

If the group G is Garside, then

- G is torsion-free [P.Dehornoy 1998]
- G is bi-automatic [P.Dehornoy 2002]
- G has word and conjugacy problem solvable
- G has finite homological dimension [P.Dehornoy and Y.Lafont 2003][R.Charney, J. Meier and K. Whittlesey 2004]

By the way, what is a Garside group?

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999.

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999.
Examples of Garside groups: Braid groups, Artin groups of finite-type.

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999.
Examples of Garside groups: Braid groups, Artin groups of finite-type.

The definition of a Garside monoid [P.Dehornoy 2002]

The monoid M is Garside if

- 1 is the unique invertible element in M.

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999. Examples of Garside groups: Braid groups, Artin groups of finite-type.

The definition of a Garside monoid [P.Dehornoy 2002]

The monoid M is Garside if

- 1 is the unique invertible element in M.
- M is left and right cancellative.

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999. Examples of Garside groups: Braid groups, Artin groups of finite-type.

The definition of a Garside monoid [P.Dehornoy 2002]

The monoid M is Garside if

- 1 is the unique invertible element in M.
- M is left and right cancellative.
- Each pair of elements in M has: left, right lcm and gcd

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999.
Examples of Garside groups: Braid groups, Artin groups of finite-type.

The definition of a Garside monoid [P.Dehornoy 2002]

The monoid M is Garside if

- 1 is the unique invertible element in M.
- M is left and right cancellative.
- Each pair of elements in M has: left, right lcm and gcd
- M has a balanced element Δ such that $\text{Div}(\Delta)$ is a finite generating set of M.

By the way, what is a Garside group?

The notion was first defined by P.Dehornoy and L.Paris in 1999. Examples of Garside groups: Braid groups, Artin groups of finite-type.

The definition of a Garside monoid [P.Dehornoy 2002]

The monoid M is Garside if

- 1 is the unique invertible element in M.
- M is left and right cancellative.
- Each pair of elements in M has: left, right lcm and gcd
- M has a balanced element Δ such that $\mathrm{Div}(\Delta)$ is a finite generating set of M.

A Garside group is the group of fractions of a Garside monoid.

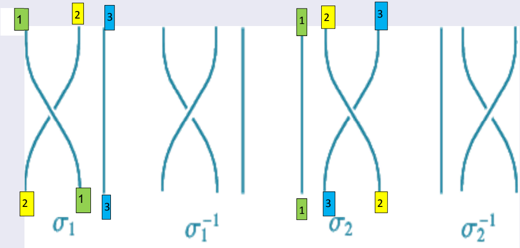
The BRAID group B_n

The BRAID group?

The BRAID group B_n

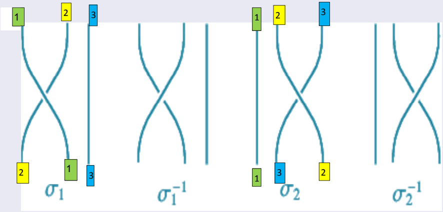
The BRAID group?

The BRAID group $B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$



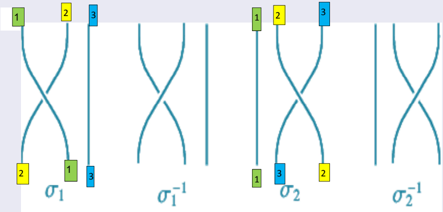
The original Coxeter group construction

$\exists$ epimorphism $B_3 \to S_3$:
$\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$



The original Coxeter group construction

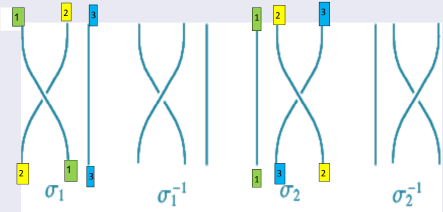
$\exists$ epimorphism $B_3 \to S_3$:
$\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$



In B_3: $\Delta = \sigma_1\sigma_2\sigma_1$

The original Coxeter group construction

$\exists$ epimorphism $B_3 \to S_3$:
$\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$

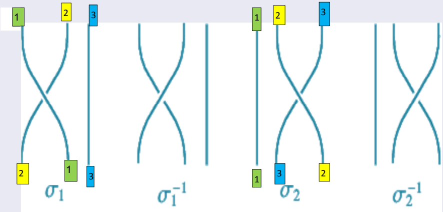


In B_3: $\Delta = \sigma_1\sigma_2\sigma_1$

$\mathrm{Div}(\Delta) = \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$

The original Coxeter group construction

$\exists$ epimorphism $B_3 \to S_3$:
$\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$

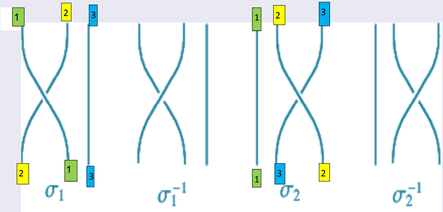


In B_3: $\Delta = \sigma_1\sigma_2\sigma_1$

$\mathrm{Div}(\Delta) = \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$
$S_3 \leftrightarrow \mathrm{Div}(\Delta)$

The original Coxeter group construction

$\exists$ epimorphism $B_3 \to S_3$:
$\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$



In B_3: $\Delta = \sigma_1\sigma_2\sigma_1$

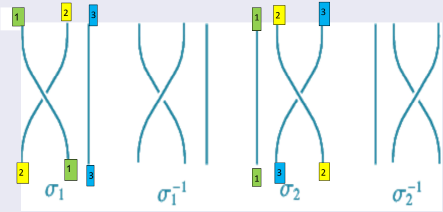
$\mathrm{Div}(\Delta) = \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$
$S_3 \leftrightarrow \mathrm{Div}(\Delta)$

The original Coxeter group

$\exists$ a short exact sequence:
$1 \to P_n \to B_n \to S_n \to 1$

The original Coxeter group construction

$\exists$ epimorphism $B_3 \to S_3$: $\sigma_1 \mapsto (1,2)$; $\sigma_2 \mapsto (2,3)$



In B_3: $\Delta = \sigma_1\sigma_2\sigma_1$

$\mathrm{Div}(\Delta) = \{\sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$

$S_3 \leftrightarrow \mathrm{Div}(\Delta)$

The original Coxeter group

$\exists$ a short exact sequence:

$1 \to P_n \to B_n \to S_n \to 1$

$\exists$ a bijection

$S_n \leftrightarrow \mathrm{Div}(\Delta)$

The motivation for the Coxeter-like finite quotient

The original Coxeter group

There exits a short exact sequence: $1 \rightarrow P_n \rightarrow B_n \rightarrow S_n \rightarrow 1$

The motivation for the Coxeter-like finite quotient

The original Coxeter group

There exits a short exact sequence: $1 \to P_n \to B_n \to S_n \to 1$
More generally, finite-type Artin groups have a finite quotient group: the finite Coxeter group.

The motivation for the Coxeter-like finite quotient

The original Coxeter group

There exits a short exact sequence: $1 \to P_n \to B_n \to S_n \to 1$
More generally, finite-type Artin groups have a finite quotient group: the finite Coxeter group.

What is so special with this finite quotient group?

The motivation for the Coxeter-like finite quotient

The original Coxeter group

There exits a short exact sequence: $1 \to P_n \to B_n \to S_n \to 1$
More generally, finite-type Artin groups have a finite quotient group: the finite Coxeter group.

What is so special with this finite quotient group?

There exits a bijection between the elements in the finite quotient group (S_n or general Coxeter) and the set $\mathrm{Div}(\Delta)$ in B_n or finite-type Artin group.

Do Coxeter-like quotient groups exist for Garside groups?

The question raised by D.Bessis

Do Coxeter-like quotient groups exist for Garside groups?

The question raised by D.Bessis

Do Garside groups admit a finite quotient that plays the same role S_n plays for B_n or the Coxeter groups for finite-type Artin groups?

Do Coxeter-like quotient groups exist for Garside groups?

The question raised by D.Bessis

Do Garside groups admit a finite quotient that plays the same role S_n plays for B_n or the Coxeter groups for finite-type Artin groups?

Our answer: yes for QYBE groups with additional condition (C)

Do Coxeter-like quotient groups exist for Garside groups?

The question raised by D.Bessis

Do Garside groups admit a finite quotient that plays the same role S_n plays for B_n or the Coxeter groups for finite-type Artin groups?

Our answer: yes for QYBE groups with additional condition (C)

Dehornoy's extension: condition (C) can be removed

The main result

Theorem (F.C and E.Godelle)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

The main result

Theorem (F.C and E.Godelle)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

- *N is a normal free abelian group of rank n*

The main result

Theorem (F.C and E.Godelle)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

- *N is a normal free abelian group of rank n*
- *There exists a bijection between W and* $\mathrm{Div}(\Delta)$

The main result

Theorem (F.C and E.Godelle)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

- *N is a normal free abelian group of rank n*
- *There exists a bijection between W and $\mathrm{Div}(\Delta)$*
- *W is a finite group or order 2^n*

The main result

Theorem (F.C and E.Godelle)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

- *N is a normal free abelian group of rank n*
- *There exists a bijection between W and $\mathrm{Div}(\Delta)$*
- *W is a finite group or order 2^n*

What is condition (C)?

Let $x_i, x_j \in X$. If $r(i,j) = (i,j)$, then $\sigma_i\sigma_j = \gamma_i\gamma_j = Id_X$.

The main result with a remark

Theorem (F.C and E.Godelle 2013)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \rightarrow N \rightarrow G \rightarrow W \rightarrow 1$
satisfying

- *N is a normal free abelian group of rank n*

The main result with a remark

Theorem (F.C and E.Godelle 2013)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$
satisfying

- *N is a normal free abelian group of rank n*

E.Jespers and J.Okninski show the existence of such a sequence

The main result with a remark

Theorem (F.C and E.Godelle 2013)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $\mid X \mid = n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

- *N is a normal free abelian group of rank n*

E.Jespers and J.Okninski show the existence of such a sequence
T.Gateva-vanova and M. Van den Bergh: G is Bieberbach

The main result with a remark

Theorem (F.C and E.Godelle 2013)

Let (X, r) be a non-degenerate, involutive and braided solution of the QYBE with structure group G and $| X |= n$. Assume (X, r) satisfies the condition (C). Then there exits a short exact sequence: $1 \to N \to G \to W \to 1$ satisfying

- *N is a normal free abelian group of rank n*

E.Jespers and J.Okninski show the existence of such a sequence T.Gateva-vanova and M. Van den Bergh: G is Bieberbach but they do not satisfy

- There exists a bijection between W and $\mathrm{Div}(\Delta)$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define S

$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1, 2, 3, 4)$
$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1, 4, 3, 2)$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define S

$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1, 2, 3, 4)$
$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1, 4, 3, 2)$

(X, r) satisfies (C): If $S(i, j) = (i, j)$, then $\sigma_i \sigma_j = \gamma_i \gamma_j = Id_X$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define S

$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1,2,3,4)$
$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1,4,3,2)$

(X, r) satisfies (C): If $S(i,j) = (i,j)$, then $\sigma_i\sigma_j = \gamma_i\gamma_j = Id_X$

The 4 trivial relations in G and in M

$x_1x_4 = x_1x_4 \qquad x_4x_1 = x_4x_1$
$x_3x_2 = x_3x_2 \qquad x_2x_3 = x_2x_3$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.

The functions that define S

$\sigma_1 = \gamma_1 = \sigma_3 = \gamma_3 = (1,2,3,4)$
$\sigma_2 = \gamma_2 = \sigma_4 = \gamma_4 = (1,4,3,2)$

(X,r) satisfies (C): If $S(i,j) = (i,j)$, then $\sigma_i\sigma_j = \gamma_i\gamma_j = Id_X$

The 4 trivial relations in G and in M

$x_1x_4 = x_1x_4 \qquad x_4x_1 = x_4x_1$

$x_3x_2 = x_3x_2 \qquad x_2x_3 = x_2x_3$

$\theta_1 = x_1x_4$, $\theta_2 = x_2x_3$, $\theta_3 = x_3x_2$, $\theta_4 = x_4x_1$ are called **frozen elements**.

The Construction of the Coxeter-like finite quotient

Definition of N: $N = \langle \theta_1, \theta_2, .., \theta_n \rangle$

The Construction of the Coxeter-like finite quotient

Definition of N: $N = \langle \theta_1, \theta_2, .., \theta_n \rangle$

- N is generated by the n frozen elements $\theta_1, \theta_2, .., \theta_n$.

The Construction of the Coxeter-like finite quotient

Definition of N: $N = \langle \theta_1, \theta_2, .., \theta_n \rangle$

- N is generated by the n frozen elements $\theta_1, \theta_2, .., \theta_n$.
- N is free abelian of rank n.

The Construction of the Coxeter-like finite quotient

Definition of N: $N = \langle \theta_1, \theta_2, .., \theta_n \rangle$

- N is generated by the n frozen elements $\theta_1, \theta_2, .., \theta_n$.
- N is free abelian of rank n.

Presentation of W

W is obtained by adding to the presentation of G the relations $\theta_i = 1$, $\forall 1 \leq i \leq n$.

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.
(X, r) satisfies the condition (C)

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.
(X, r) satisfies the condition (C)

The 4 frozen elements

$\theta_1 = x_1x_4,\ \theta_2 = x_2x_3,\ \theta_3 = x_3x_2,\ \theta_4 = x_4x_1$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.
(X, r) satisfies the condition (C)

The 4 frozen elements

$\theta_1 = x_1x_4,\ \theta_2 = x_2x_3,\ \theta_3 = x_3x_2,\ \theta_4 = x_4x_1$

The normal free abelian subgroup N

$N = \langle \theta_1, \theta_2, \theta_3, \theta_4 \rangle = \langle x_1x_4,\ x_2x_3,\ x_3x_2,\ x_4x_1 \rangle$

The example

Let $X = \{x_1, x_2, x_3, x_4\}$.
(X, r) satisfies the condition (C)

The 4 frozen elements

$\theta_1 = x_1x_4,\ \theta_2 = x_2x_3,\ \theta_3 = x_3x_2,\ \theta_4 = x_4x_1$

The normal free abelian subgroup N

$N = \langle \theta_1, \theta_2, \theta_3, \theta_4 \rangle = \langle x_1x_4,\ x_2x_3,\ x_3x_2,\ x_4x_1 \rangle$

The Coxeter-like group W

$W = \langle\ G\ \mid x_4 = x_1^{-1},\ x_3 = x_2^{-1}\ \rangle$

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh, \forall f, g, h \in G$.

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh,\ \forall f, g, h \in G$.

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk,\ \forall f, g, h, k \in G$.

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh, \forall f, g, h \in G$.

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk, \forall f, g, h, k \in G$.

Examples of bi-orderable and left-orderable groups

Bi-orderable: free groups,

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh, \forall f, g, h \in G.$

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk, \forall f, g, h, k \in G.$

Examples of bi-orderable and left-orderable groups

Bi-orderable: free groups,torsion-free abelian groups,

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh, \forall f, g, h \in G.$

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk, \forall f, g, h, k \in G.$

Examples of bi-orderable and left-orderable groups

Bi-orderable: free groups,torsion-free abelian groups,pure braid groups,

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh, \forall f, g, h \in G$.

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk, \forall f, g, h, k \in G$.

Examples of bi-orderable and left-orderable groups

Bi-orderable: free groups,torsion-free abelian groups,pure braid groups, f.g of surfaces except the Klein bottle group and the projective plane's group
Left-orderable: knot groups,

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh$, $\forall f, g, h \in G$.

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk$, $\forall f, g, h, k \in G$.

Examples of bi-orderable and left-orderable groups

Bi-orderable: free groups,torsion-free abelian groups,pure braid groups, f.g of surfaces except the Klein bottle group and the projective plane's group
Left-orderable: knot groups, braid groups,

Orderability of groups

A group G is *left-orderable*

if there exists a strict total ordering $\prec$ of its elements which is invariant under left multiplication:
$g \prec h \Longrightarrow fg \prec fh, \forall f, g, h \in G$.

G is *bi-orderable*

if $\prec$ is invariant under left and right multiplication:
$g \prec h \Longrightarrow fgk \prec fhk, \forall f, g, h, k \in G$.

Examples of bi-orderable and left-orderable groups

Bi-orderable: free groups,torsion-free abelian groups,pure braid groups, f.g of surfaces except the Klein bottle group and the projective plane's group
Left-orderable: knot groups, braid groups, Homeo$^+(\mathbb{R})$

Are all the Garside groups left-orderable?

Are all the Garside groups left-orderable?

Question from book *Ordering braids*
of P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest

Are all the Garside groups left-orderable?

Are all the Garside groups left-orderable?

Question from book *Ordering braids*
of P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest

The short answer is: Not necessarily!!

Are all the Garside groups left-orderable?

Are all the Garside groups left-orderable?

Question from book *Ordering braids*
of P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest

The short answer is: Not necessarily!!

Left orders in Garside group (IJAC 2016)

In the book of E. Jespers and I. Okninski (2007):

Are all the Garside groups left-orderable?

Are all the Garside groups left-orderable?

Question from book *Ordering braids*
of P. Dehornoy, I. Dynnikov, D. Rolfsen and B. Wiest

The short answer is: Not necessarily!!

Left orders in Garside group (IJAC 2016)

In the book of E. Jespers and I. Okninski (2007):

There exist Garside groups that do not satisfy the unique product property.

Are all the Garside groups left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

Are all the Garside groups left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

For multipermutations (retractable) solutions, we show

Their structure group satisfies a property stronger than locally indicable ("almost" bi-orderable)

Are all the Garside groups left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

For multipermutations (retractable) solutions, we show

Their structure group satisfies a property stronger than locally indicable (“almost” bi-orderable)

In our paper, we asked whether there exist structure groups of non-retractable solutions that are left-orderable.

Are all the Garside groups left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

For multipermutations (retractable) solutions, we show

Their structure group satisfies a property stronger than locally indicable ("almost" bi-orderable)

In our paper, we asked whether there exist structure groups of non-retractable solutions that are left-orderable.

The answer is: No!
Bachiller-Cedo-Vendramin- 2017

So what if a group is left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

So what if a group is left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

A group G satisfies *the unique product property*, if for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$.

So what if a group is left-orderable?

Bi-orderable ⇒ Locally indicable ⇒ Left-orderable ⇒ Unique product ⇒ Torsion-free

A group G satisfies *the unique product property*, if for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$.

For a torsion free group

Unique product ⇒ Kaplansky's Unit conjecture satisfied: the units in the group algebra are trivial

So what if a group is left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

A group G satisfies *the unique product property*, if for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$.

For a torsion free group

Unique product $\Rightarrow$ Kaplansky's Unit conjecture satisfied $\Rightarrow$ Kaplansky's Zero-divisor conjecture satisfied

So what if a group is left-orderable?

Bi-orderable ⇒ Locally indicable ⇒ Left-orderable ⇒ Unique product ⇒ Torsion-free

A group G satisfies *the unique product property*, if for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$.

For a torsion free group

Unique product ⇒ Kaplansky's Unit conjecture satisfied ⇒ Kaplansky's Zero-divisor conjecture satisfied: there are no zero divisors in the group algebra

So what if a group is left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

A group G satisfies *the unique product property*, if for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$.

For a torsion free group

Unique product $\Rightarrow$ Kaplansky's Unit conjecture satisfied $\Rightarrow$ Kaplansky's Zero-divisor conjecture satisfied $\Rightarrow$ Kaplansky's Idempotent conjecture satisfied

So what if a group is left-orderable?

Bi-orderable $\Rightarrow$ Locally indicable $\Rightarrow$ Left-orderable $\Rightarrow$ Unique product $\Rightarrow$ Torsion-free

A group G satisfies *the unique product property*, if for any finite subsets $A, B \subseteq G$, there exists at least one element $x \in AB$ that can be uniquely written as $x = ab$, with $a \in A$ and $b \in B$.

For a torsion free group

Unique product $\Rightarrow$ Kaplansky's Unit conjecture satisfied $\Rightarrow$ Kaplansky's Zero-divisor conjecture satisfied $\Rightarrow$ Kaplansky's Idempotent conjecture satisfied: there are no non-trivial idempotents in the group algebra

Some remarks to conclude

- G is a Bieberbach group (T. Gateva-Ivanova and M. Van den Bergh, P. Etingof et al.) i.e. it is a torsion free crystallographic group.

Some remarks to conclude

Some remarks to conclude

- G is a Bieberbach group (T. Gateva-Ivanova and M. Van den Bergh, P. Etingof et al.) i.e. it is a torsion free crystallographic group.
- Bieberbach groups satisfy Kaplansky's zero divisor conjecture, as it holds for all torsion-free finite-by-solvable groups (P.H. Kropholler, P.A. Linnell, and J.A. Moody).

Some remarks to conclude

Some remarks to conclude

- G is a Bieberbach group (T. Gateva-Ivanova and M. Van den Bergh, P. Etingof et al.) i.e. it is a torsion free crystallographic group.
- Bieberbach groups satisfy Kaplansky's zero divisor conjecture, as it holds for all torsion-free finite-by-solvable groups (P.H. Kropholler, P.A. Linnell, and J.A. Moody).
- B_n satisfy the zero divisor conjecture, as they are left-orderable (P. Dehornoy).

The original question can be replaced by:

Question: does a Garside group satisfy Kaplansky's zero divisor conjecture?

The end

Thank you!