Set-theoretic solutions of the pentagon equation

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Noncommutative and non-associative structures, braces and applications

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Motivation

My interest in the pentagon equation starts from the following paper

A. Van Daele, S. Van Keer, *The Yang-Baxter equation and pentagon equation*, Compos. Math. **91** (1994), 201–221.

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In this talk I will present some classic results about solutions of the pentagon equation. Moreover, I will deal with set-theoretical solutions, showing both old and some new results that are in the paper

F. Catino, M. Mazzotta, M.M. Miccoli, *The set-theoretic solutions of the pentagon equation*, work in progress.

Solutions of the pentagon equation

Definition

Let V be a vector space over a field F. A linear map $S : V \otimes V \to V \otimes V$ is said to be a solution of the pentagon equation if

$$S_{12}S_{13}S_{23} = S_{23}S_{12}$$

where the map $S_{ij} : V \otimes V \otimes V \to V \otimes V \otimes V$ acting as S on the (i, j) tensor factor and as the identity on the remaining factor.

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Solutions of the pentagon equation appear in various contexts and with different terminology.

Fusion operators

For istance in
R. Street, *Fusion operators and Cocycloids in Monomial Categories*, Appl. Categor. Struct. **6** (1998), 177–191

a solution of the pentagon equation is said to be a *fusion operator*.

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a solution of the pentagon equation is said to be a *fusion operator*.

Example

Let B be a bialgebra with product $m : B \otimes B \longrightarrow B$ and coproduct $\Delta : B \longrightarrow B \otimes B$. Then

$$S := (id_B \otimes m)(\Delta \otimes id_B)$$

is a solution of the pentagon equation (or fusion operator).

Example

Let B be a Hopf algebra with product $m : B \otimes B \longrightarrow B$, coproduct $\Delta : B \longrightarrow B \otimes B$ and antipode $\nu : B \longrightarrow B$. Then S is invertible and the inverse is given by

$$S^{-1} = (1_A \otimes m)(1_A \otimes \nu \otimes 1_A)(\Delta \otimes 1_A).$$

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$$S^{-1} = (1_A \otimes m)(1_A \otimes \nu \otimes 1_A)(\Delta \otimes 1_A).$$

Note that S^{-1} is a solution of the reversed pentagon equation

$$S_{23}S_{13}S_{12} = S_{12}S_{23}.$$

In

G. Militaru, *The Hopf modules category and the Hopf equation*, Comm. Algebra **10** (1998), 3071–3097

this equation is called Hopf equation.

Multiplicative operators

Let $\mathcal{H}$ be a Hilbert space. A unitary operator acting on $\mathcal{H} \otimes \mathcal{H}$ satisfying the pentagon equation, has been termed *multiplicative*.

These operators were introduced by Enok and Schwartz in the study of duality theory for Hopf-von Neumann algebras.
[M. Enok, J.-M Schwartz, Kac Algebras and Duality of Locally Compact Groups, *Springer-Verlag, Berlin* (1992)].

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Example (Kac-Takesaki operator)

Let G be a locally compact group. Fix a left Haar measure on G and let $\mathcal{H} = L^2(G)$ denote the Hilbert space of square integrable complex functions on G. Then the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}$ is (isomorphic to) the Hilbert space $L^2(G \times G)$. Let S_G be the unitary operator acting on $\mathcal{H} \otimes \mathcal{H}$ defined by

$$(S_G\varphi)(x, y) = \varphi(xy, y)$$

for all $\varphi \in \mathcal{H}$ and $x, y \in G$. Then S_G is multiplicative, that is a solution of the pentagon equation.

An abstract way

Kashaev and Sergeev watch this kind of operators in an abstract way.
[R.M. Kashaev, S.M. Sergeev, *On Pentagon, Ten-Term and Tetrahedrom Relations*, Commun. Math. Phys. **1995** (1998), 309–319].

Example

Let G be a group. Let $\mathbb{C}^G$ denote the vector space over the complex field $\mathbb{C}$ of the functions from G to $\mathbb{C}$. The operator S_G on $\mathbb{C}^{G\times G}$ defined by

$$(S_G\varphi)(x, y) = \varphi(xy, y),$$

for all $\varphi \in \mathbb{C}^{G\times G}$ and $x, y \in G$, is a solution of the pentagon equation.

Set-theoretic solutions of the pentagon equation

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Definition

Let M be a set. A set-theoretic solution of the pentagon equation on M is a map $s : M \times M \longrightarrow M \times M$ which satisfy the "reversed" pentagon equation

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23}$$

where $s_{12} = s \times id_M$, $s_{23} = id_M \times s$ and $s_{13} = (id_M \times \tau)s_{12}(id_M \times \tau)$ with τ the flip map.

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Example

Let G be a group. The map $s : G \times G \longrightarrow G \times G, (x, y) \mapsto (xy, y)$ is a set-theoretic solution of the pentagon equation.

Note that the flip map τ is not a set-theoretic solution of the pentagon equation if $|M| > 1$.

A bridge

Proposition

Let M be a set and F be a field. If $V := F^M$, then the tensor product $V \otimes V$ is isomorphic to $F^{M\times M}$. Let $s : M \times M \to M \times M$ and define the operator S on $V \otimes V$ by

$$(S\varphi)(x,y) = \varphi(s(x,y))$$

for all $\varphi \in F^{M\times M}$ and $x, y \in M$.
Then S is a solution of the pentagon equation if and only if s is a set-theoretic solution of the pentagon equation.

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Then S is a solution of the pentagon equation if and only if s is a set-theoretic solution of the pentagon equation.

Example (Kac-Takesaki solution)

If M is a group, then the map $s : M \times M \to M \times M, (x, y) \mapsto (xy, y)$ is a set-theoretic solution. So, the operator S defined by

$$(S\varphi)(x, y) = \varphi(xy, y)$$

for all $\varphi \in F^{M\times M}$ and $x, y \in M$, is a solution of the pentagon equation.

Another version of Kac-Takesaki solution

Example

If M is a group, then the map $s : M \times M \to M \times M, (x, y) \mapsto (x, yx^{-1})$ is a set-theoretic solution. So, the operator S defined by

$$(S\varphi)(x, y) = \varphi(x, yx^{-1})$$

for all $\varphi \in F^{M\times M}$ and $x, y \in M$, is a solution of the pentagon equation.

Set-theoretic solutions of the reversed pentagon equation

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Definition

Let M be a set. A set-theoretic solution of the reversed pentagon equation on M is a map $s : M \times M \longrightarrow M \times M$ which satisfies the condition

$$s_{12}\, s_{13}\, s_{23} = s_{23}\, s_{12}$$

where $s_{12} = s \times id_M$, $s_{23} = id_M \times s$ and $s_{13} = (id_M \times \tau)s_{12}(id_M \times \tau)$ with τ the flip map.

Remark

A map $s : M \times M \longrightarrow M \times M$ is a set-theoretic solution of the pentagon equation if and only if $\tau s \tau$ is a set-theoretic solution of the reversed pentagon equation.
Moreover, if s is invertible, then s^{-1} is a set-theoretic solution of the reversed pentagon equation on M.

Isomorphic solutions

Definition

Let M, N be two sets, s be a solution on M and r be a solution on N. Then s and r are called isomorphic if there exists a bijective map $\alpha : M \to N$ such that

$$s = (\alpha^{-1} \times \alpha^{-1}) r (\alpha \times \alpha).$$

Example

Let G be a group. Then the solutions $r, s : G \times G \to G \times G$ defined by

$$r(x.y) = (yx, y), \quad s(x.y) = (xy, y)$$

are isomorphic by $\alpha : G \to G,\ x \mapsto x^{-1}$.

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A challenging question:
are the solutions $s(x, y) = (xy, y)$ and $s^{op}(x, y) = (x, yx^{-1})$ related to two versions of Kac-Takesaki operator isomorphic?

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I will answer later.

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are the solutions $s(x, y) = (xy, y)$ and $s^{op}(x, y) = (x, yx^{-1})$ related to two versions of Kac-Takesaki operator isomorphic?
I will answer later. Why wait? No!

Related structures

For a map $s : M \times M \to M \times M$ define binary operations $\cdot$ and $*$ as

$$s(x,y) = (x \cdot y \;,\; x * y).$$

Lemma

Let M be a set. A map $s : M \times M \to M \times M$ is a solution of the pentagon equation if and only if the following conditions hold

(1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(2) $(x * y) \cdot ((x \cdot y) * z) = x * (y \cdot z)$

(3) $(x * y) * ((x \cdot y) * z) = y * z$

for all $x, y, z \in M$.
Moreover, s is invertible if and only if for any pair $(x,y) \in M \times M$ there exists a unique pair $(u,z) \in M \times M$ such that

(4) $u \cdot z = x, \quad u * z = y.$

A question

Kashaev and Reshetikhin in *Symmetrically Factorizable Groups and Set-theoretical Solutions of the Pentagon Equation*, Contemp. Math. **433** (2007), 267–279 (2007)
noted that assuming $(M, \cdot)$ is a group greatly limits the operation $*$.

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Corollary

Let $\cdot$ and $*$ be a pair of operations on a set M satisfying the conditions (1)–(4) of the Lemma. If the operation $\cdot$ defines a group structure on M, then $x * y = y$ for all $x, y \in M$.
So, if M is a group, then the only invertible solution s of the pentagon equation on M with $x \cdot y = xy$, for all $x, y \in M$, is given by $s(x, y) = (xy, y)$.

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If $(M, \cdot)$ is a group, then a solution of the pentagon equation on M is given by $s(x, y) = (x \cdot y, 1)$ for all $x, y \in M$.
Actually, we are not able to obtain all solutions when the operation $\cdot$ is a group operation.

Some examples

Example

Let $(M, \cdot)$ *be a semigroup, and let* α *be an endomorphism of* $(M, \cdot)$ *such that* $\alpha^2 = \alpha$. *Define* $x * y := \alpha(y)$, *for all* $x, y \in M$, *then the pair of operations* $\cdot$ *and* $*$ *satisfied the conditions (1)-(3) of Lemma. Hence the map* $s : M \times M \to M \times M$ *given by*

$$s(x, y) = (xy, \alpha(y))$$

is a solution of the pentagon equation.

Interesting invertible set-theoretic solutions can be obtained assuming M is a closed subset of a group G
[R.M. Kashaev, S.M. Sergeev, *On Pentagon, Ten-Term and Tetrahedrom Relations*, Commun. Math. Phys. **1995** (1998), 309–319].

Proposition

Let M be a closed subset of a group $(G, \cdot)$, and let $\lambda, \mu : M \to G$ be maps such that

$$x * y = \mu(x)^{-1}\mu(xy) \in M, \qquad \mu(x * y) = \lambda(x)\mu(y),$$

for all $x, y \in M$. Then the pair of operations $\cdot$ and $*$ satisfies the conditions (1)-(4) of Lemma. If furthermore $1 \in M$, then $x * y = y$ is the only possibility for the operation $*$.
Consequently, the map $s : M \times M \to M \times M$ given by

$$s(x, y) = (xy,\ \mu(x)^{-1}\mu(xy))$$

for all $x, y \in M$, is a set-theoretic solution of the pentagon equation.

Example

Let $M =]0,1[\subseteq \mathbb{R}^*$ *be the open unit interval with the dot-mapping* $\cdot$ *given by the multiplication in* $\mathbb{R}$*. Set*

$$\mu(x) = \frac{x}{1-x} \quad \text{and} \quad \lambda(x) = 1 - x,$$

for all $x \in M$*. Then*

$$s(x,y) = (xy, \frac{(1-x)y}{1-xy})$$

is a solution on M*.*

Examples of Zakrzewski

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S. Zakrzewski, *Poisson Lie Groups and Pentagonal Transformations*, Lett. Math. Phys. **24** (1992), 13–19.

Example

Let G be a group and A, B its subgroups such that $G = AB$ and $A \cap B = \{1\}$. Then for every $x \in G$ there exists a unique couple $(a, b) \in A \times B$ such that $x = ab$.
Let $p_1 : G \to A$ and $p_2 : G \to B$ be maps such that $x = p_1(x)p_2(x)$, for every $x \in G$.

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Let $p_1 : G \to A$ and $p_2 : G \to B$ be maps such that $x = p_1(x)p_2(x)$, for every $x \in G$.
Then the map $s : G \times G \longrightarrow G \times G$ defined by

$$s(x, y) = (p_2(yp_1(x)^{-1})x, yp_1(x)^{-1})$$

for all $x, y \in G$, is a solution of the pentagon equation.

Examples of Baaj and Skandalis

The following example is slightly different from Zakrzewski's one.

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S. Baaj, G. Skandalis, *Unitaries multiplicatifs et dualité pour le produits croisés de C*-algèbres*, Ann. Sci. Éc. Norm. Sup. (4) **26** (1993), 425–488

Example

Let G be a group and A, B its subgroups such that $G = AB$ and $A \cap B = \{1\}$. Let $p_1 : G \to A$ e $p_2 : G \to B$ maps such that $x = p_1(x)p_2(x)$, for every $x \in G$. Then the map $s : G \times G \longrightarrow G \times G$ defined by

$$s(x, y) = (xp_1(p_2(x)^{-1}y),\ p_2(x)^{-1}y)$$

for all $x, y \in G$, is a solution of the pentagon equation.

Opposite operators

Following T. Timmerman, An invitation to Quantum Groups and Duality. *Europan Math. Soc.* (2008) we give the

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Definition

Let V be a vector space over a field F and, Σ be the flip map on $V \otimes V$. If $S : V \otimes V \to V \otimes V$ is an invertible operator, then

$$S^{op} := \Sigma S^{-1} \Sigma$$

is the opposite operator of S.

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$$S^{op} := \Sigma S^{-1} \Sigma$$

is the opposite operator of S.

Example (Kac-Takesaki operators)

If M is a group and $(S\varphi)(x, y) = \varphi(xy, y)$, for all $\varphi \in \mathbb{C}^{M \times M}$ and $x, y \in M$, is the Kac-Takesaki operator, then

$$(S^{op}\varphi)(x, y) = \varphi(x, yx^{-1})$$

is the opposite operator of S.

Opposite solutions

Definition

Let M be a set and s be an invertible solution on M. Then

$$s^{op} := \tau s^{-1} \tau$$

is the opposite solution of s.

Example

Let G be a group. The following maps $s, r : G \times G \longrightarrow G \times G$ defined by

1) $s(x, y) = (xy,\ y)$, $r(x, y) = (x,\ yx^{-1})$,
2) $s(x, y) = (yx,\ y)$, $r(x, y) = (x,\ x^{-1}y)$

are opposite solutions of the pentagon equation on G.

Commutative and cocommutative solutions

Following Baaj and Skandalis [*Unitaries multiplicatifs et dualité pour le produits croisés de C*-algèbres*, Ann. Sci. Éc. Norm. Sup. (4) **26** (1993), 425–488]

Definition

Let M be a set and $s : M \times M \to M \times M$ be a solution. Then

(1) *s is called commutative if $s_{13}s_{23} = s_{23}s_{13}$;*

(2) *s is called cocommutative if $s_{13}s_{12} = s_{12}s_{13}$.*

Example

Let G be a group.
Then the solution given by $s(x, y) = (xy, y)$, for all $x, y \in G$, is commutative.
Instead, the solution given by $s^{op}(x, y) = (x, yx^{-1})$, for all $x, y \in G$, is cocommutative.

Remark

If s is an invertible solution on a set M, then s is commutative (respectively cocommutative) if and only if s^{op} is cocommutative (respectively commutative).

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If s is an invertible solution on a set M, then s is commutative (respectively cocommutative) if and only if s^{op} is cocommutative (respectively commutative).

Example

Let M be a set and $f, g : M \to M$ be two maps such that $f^2 = f$, $g^2 = g$ and $fg = gf$. Then the map

$$s : M \times M \to M \times M, \quad (x, y) \mapsto (f(x), g(y))$$

is a solution both commutative and cocommutative.

Quasi-linear solutions

Quasi-linear solutions

L. Jiang, M. Liu, *On set-theoretical solution of the pentagon equation*, Adv. Math. (China) **34** (2005), 331–337

Definition

Let G be a (additive) group. A map $s : G \times G \to G \times G$ is called quasi-linear if

$$s(x, y) = (A(x) + B(y),\ C(x) + D(y))$$

where $A, B, C, D \in End(G)$. If G is abelian, s is called linear.

Proposition

Let G be a group. Then a quasi-linear map s is a solution of the pentagon equation if and only if

(1) $A = A^2$	(2) $B = B^2$	(3) $D^2 = D$
(4) $[A, B] = 0$	(5) $BCB = [-D, A]$	(6) $[B, D] = 0$
(7) $C^2 = -DCA$	(8) $AC = C - BCA$	(9) $CD = C - DCB$.

Quasi-affine solutions

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Definition

Let G be a group. A map $s : G \times G \to G \times G$ is called quasi-affine if

$$s(x, y) = (A(x) + B(y) + u,\ C(x) + D(y) + v)$$

where $A, B, C, D \in End(G)$ and $u, v \in G$. If G is abelian, s is called affine.

Proposition

Let G be a group. Then a quasi-affine map s is a solution of the pentagon equation if and only if conditions (1)-(9) of above Proposition are satisfied, C and D are invertible and $u = C^{-1}D^{-1}(-C - D)(v)$.

P-involutive solutions

L. Jiang, M. Liu also characterized all P-involutive solutions.

Definition

Let G be a (additive) finite group and $P \in End(G)$. A function $s : G \times G \longrightarrow G \times G$ is called P-involutive if

$$(\sigma s)^2 = P \times P.$$

where $\sigma : G \times G \longrightarrow G \times G, (x, y) \mapsto (-x, y)$.

Semisymmetric solutions

R. Kashaev in *Full noncommutative discrete Liouville equation*, RIMS (2011), 89–98 considers this type of solutions.

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Definition

Let M be a set. A solution $s : M \times M \longrightarrow M \times M$ is called semisymmetric if there exists a map $\alpha : M \longrightarrow M$ such that

$$\alpha^3 = id_M, \quad s\tau(\alpha \times id_M)s = \alpha \times \alpha$$

where τ is the flip map.

Semisymmetric solutions

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where τ is the flip map.

He gives solutions using *groups with addition*.

Definition

A group G is called group with addition if it is provided with an associative and commutative binary operation, called addition, with respect to which the group multiplication is distributive.

- The set of positive real numbers is naturally a group with addition as well as its subgroup of positive rationals.
- The group of integers $\mathbb{Z}$ is also a group with addition where the addition is the maximum operation $max(m, n)$.
- An example of a non Abelian group with addition is given by the group of upper-triangular real two-by-two matrices with positive reals on the diagonal. The addition here is given by the usual matrix addition.

Proposition

Let G be a group with addition and $c \in G$ a central element. Then there exists a set-theoretic semisymmetric solution $s(x, y) = (x \cdot y, x * y)$ on $G \times G$ where

$$x \cdot y = (x_1, x_2)(y_1, y_2) = (x_1 y_1,\ x_1 y_2 + x_2)$$

and

$$x * y = ((1 + y_2 x_2^{-1} x_1)^{-1} y_1,\ (1 + y_2 x_2^{-1} x_1)^{-1} y_2 x_2^{-1}),$$

with $\alpha(x_1, x_2) = (c x_1^{-1} x_2,\ x_1^{-1})$.

THANK YOU!