Indecomposable simple left cycle sets

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Definition (Rump, 2005)

A *left cycle set* is a non-empty set X with a binary operation $\cdot$ such that the left multiplication $\sigma_x : X \longrightarrow X,\ y \longmapsto x \cdot y$ is bijective, and the equation

$$(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z) \tag{1}$$

holds for all $x, y, z \in X$.
A left cycle sets is said *non-degenerate* if $\mathfrak{q} : X \to X,\ x \mapsto x \cdot x$ is bijective.

Rump (2005) showed that there is a bijective correspondence between the involutive set-theoretic solutions of the Yang-Baxter equation and the left non-degenerate cycle sets.
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A homomorphism $p : X \longrightarrow Y$ between the left cycle sets X to the left cycle set Y is said a *covering map* if it is surjective and all the sets $p^{-1}(y)$, where $y \in Y$, have the same cardinality.

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If Y is a left cycle set and S a non-empty set, then $\alpha : Y \times Y \times S \longrightarrow Sym(S)$, $(i, j, s) \mapsto \alpha_{i,j}(s, -)$ is a *dynamical cocycle* if

$$\alpha_{i \cdot j, i \cdot k}(\alpha_{i,j}(r, s), \alpha_{i,k}(r, t)) = \alpha_{j \cdot i, j \cdot k}(\alpha_{j,i}(s, r), \alpha_{j,k}(s, t)) \qquad (2)$$

for all $i, j, k \in Y$, $r, s, t \in S$.

Theorem (Vendramin,2016)

If a finite left cycle set X admits a covering map p into the left cycle set Y, then there exists a non-empty set S and a dynamical cocycle α such that X is isomorphic to the left cycle set $S \times_\alpha Y := (S \times Y, \cdot)$, where

$$(s, i) \cdot (t, j) := (\alpha_{ij}(s, t), i \cdot j) \qquad (3)$$

for all $i, j \in Y$, $s, t \in S$. The left cycle set $S \times_\alpha Y$ is said to be the dynamical extension of Y by α.

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In order to classify all the left cycle sets, focus our attention on the covering map p between the left cycle set X to the left cycle set Y:

- if the covering map is not trivial, by Vendramin's result, we can study the left cycle set X, analyizing the left cycle set Y of less cardinality and the dynamical cocycles;
- it is possible to iterate this process until the obtained left cycle set has all the covering maps trivial.
  This is the case of the simple left cycle sets.

Definition (Vendramin,2016)

Let X, Y be left cycle sets. A left cycle set X, $|X| > 1$, is said to be *simple* if every covering map $p : X \longrightarrow Y$, is trivial.

The indecomposable one play a key-role in the classification and in order to costruct new examples of left cycle sets:

Definition (Rump, 2005)

A non-degenerate left cycle set X is said to be *decomposable* if it is an union of two disjoint non-empty subsets $X = Y \cup Z$, such that $\sigma_x(Y) \subseteq Y$ for all $x \in Y$ and $\sigma_x(Z) \subseteq (Z)$, for all $x \in Z$. Otherwise it is called *indecomposable*.

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Therefore, the kwnoledge of simple finite left cycle sets allows us to study all the indecomposable finite left cycle sets as dynamical extensions of simple indecomposable left cycle sets.

In other words: starting from a simple indecomposable left cycle set, via dynamical cocycle, it is possible to contruct a new left cycle set as dynamical extensions and then verify its indecomposability.

Example (Vendramin, 2016)

Every left cycle set having a prime number of elements is simple.

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Let X be the left cycle set given by

$\cdot$	1	2	3	4
1	1	3	2	4
2	4	2	3	1
3	3	1	4	2
4	2	4	1	3

Then X is simple.

Vendramin prove that if there exists a non-trivial covering map from X to another left cycle sets Y, then Y has cardinality 2.
Analyzing the only two possible cases, he arrives to a contraddiction.

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Problem

Let X be a set of 8 elements. X is an indecomposable left cycle set with the operation given by $\sigma_i(j) := i \cdot j$ where

$$\sigma_1 := (15382647) \quad \sigma_2 := (15)(26)(3847)$$

$$\sigma_3 := (16482537) \quad \sigma_4 := (1625)(37)(48)$$

$$\sigma_5 := (17462835) \quad \sigma_6 := (15)(26)(3748)$$

$$\sigma_7 := (17352846) \quad \sigma_8 := (1526)(37)(48)$$

Is X simple?

If the cardinality of the left cycle set is more than 4 it is not easy to verify by definition its semplicity.

For this reason we give other results that help us in this direction.

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Let $\mathcal{G}(X)$ the subgroup of $Sym(X)$ generated by the left moltiplication σ_x, for every $x \in X$, we call $\mathcal{G}(X)$ the *permutation group* associated to the left cycle set X.

Proposition

If the permutation group $\mathcal{G}(X)$ is primitive on X then X is simple.

Note that the converse of the previous proposition in general is not true.

Example

Let X be the cycle set given by

$$\begin{array}{c|cccc} \cdot & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 3 & 2 & 4 \\ 2 & 4 & 2 & 3 & 1 \\ 3 & 3 & 1 & 4 & 2 \\ 4 & 2 & 4 & 1 & 3 \end{array}$$

The left cycle set is simple but $\mathcal{G}(X)$ is not primitive since $\{\{1,4\},\{2,3\}\}$ is a system block.

Theorem (M. Castelli, F.Catino, G.P., in preparation)

Let $(X, \cdot)$ be an indecomposable finite left cycle set. Then X is not simple if and only if $\mathcal{G}(X)$ has an imprimitivity block system $\{\Delta_x\}_{x\in X}$ such that for all $x, x' \in X$ if $\Delta_x = \Delta_{x'}$ then $\sigma^*_x = \sigma^*_{x'}$.

Where, for every $g \in \mathcal{G}(X)$, we indicate by g^* the induced permutation on $\{\Delta_x\}_{x\in X}$, given by

$$g(\Delta_x) := \Delta_{g(x)}$$

for every $x \in X$.

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The only imprimitive block system is $\{\{1,4\},\{2,3\}\}$ but $\sigma^*_1 = id_{\{\Delta_x\}_x}$ and $\sigma^*_4 = (\Delta_1\Delta_2)$. Then X is simple.

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Then X is not simple since $\{\{1,2,3,4\},\{5,6,7,8\}\}$ is an imprimitive block system and $\sigma_x^* = (\Delta_1 \Delta_5)$ for every $x \in X$.

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Let X be a set of 8 elements. X is an indecomposable left cycle set with the operation given by $\sigma_i(j) := i \cdot j$ where

$$\sigma_1 := (14)(28)(35) \quad \sigma_2 := (1342)(57)(68)$$

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$$\sigma_5 := (17)(46)(58) \quad \sigma_6 := (13)(24)(5786)$$

$$\sigma_7 := (28)(35)(67) \quad \sigma_8 := (13)(24)(5678).$$

There are two imprimitive blocks systems:

- $S_1 := \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\}$ and $\sigma_1^* = (\Delta_2\Delta_5) \neq \sigma_4^* = (\Delta_1\Delta_2)(\Delta_5\Delta_6)$;
- $S_2 := \{\{1,4,6,7\},\{3,2,5,8\}\}$ and $\sigma_1^* = id_{S_2} \neq \sigma_4^* = (\Delta_1\Delta_2)$.

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THANKS FOR YOUR ATTENTION!