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On Some Questions Related to the Köthe's Problem

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Notation

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- R stands for an associative (usually unital) ring.
- A subset S of R is nil if every element of S is nilpotent.
- $J(R), N(R)$ indicate the Jacobson and upper nil radicals of a ring R, respectively.

Köthe's Problem

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3. *If R is a nil ring, then so is the matrix ring $M_n(R)$, for any $n \in \mathbb{N}$.*

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2. *(J.Krempa (1972) and A.D.Sands (1973)) If R is a nil ring, then so is the ring $M_2(R)$ of 2×2 matrices over R.*
3. *If R is a nil ring, then so is the matrix ring $M_n(R)$, for any $n \in \mathbb{N}$.*
4. *(J.Krempa (1972)) For any ring R, $J(R[x]) = N(R)[x]$.*

Köthe's Problem

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2. Question: Does R nil imply that $R[x]$ is nil?

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1. $J(R[x]) = N[x]$ for some nil ideal N of R.
2. Question: Does R nil imply that $R[x]$ is nil?
3. Question: Does R nil imply that $R[x]$ is a Jacobson radical ring?

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Theorem

1. *(J.Krempa(1972)) If R is an algebra over uncountable field, the above question (2) has positive answer.*

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1. $J(R[x]) = N[x]$ for some nil ideal N of R.
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3. Question: Does R nil imply that $R[x]$ is a Jacobson radical ring?

Theorem

1. *(J.Krempa(1972)) If R is an algebra over uncountable field, the above question (2) has positive answer.*
2. *(A.Smoktunowicz (2000)) The question (2) has negative answer for algebras over countable fields.*

Köthe's Problem

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Theorem (Krempa (1972))

The following conditions are equivalent:

1. *Köthe's Problem has a positive solution.*
2. *Köthe's Problem has a positive solution for algebras over fields.*

Definitions

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Definition

1. (W.K.Nicholson (1977)) An element $a \in R$ is clean if $a = e + u$, for an idempotent e and a unit u of R.

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2. (A. J. Diesl (2013)) An element $a \in R$ is nil clean if $a = e + n$, for an idempotent e and a nilpotent n of R.
3. A ring R is (nil) clean if every element of R is (nil) clean.

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Remark

- The polynomial ring $R[x]$ is never clean.

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Remark

- The polynomial ring $R[x]$ is never clean.
- If R is clean then $R[[x]]$ is clean.

Motivation

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Remark

$Cl(R[[x]]) = Cl(R) + xR[[x]]$

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Question(P. Kanwar, A. Leroy, J.M.)

What are the necessary and sufficient conditions, in terms of properties of R, for $Cl(R[x])$ to be a subring of $R[x]$.

Results

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Proposition (P. Kanwar, A. Leroy, J.M.)

Suppose that $Cl(R[x])$ is a subring of $R[x]$. Then:

(i) *$Cl(R)$ is a subring of R;*

Results

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Proposition (P. Kanwar, A. Leroy, J.M.)

Suppose that $Cl(R[x])$ is a subring of $R[x]$. Then:

(i) $Cl(R)$ *is a subring of* R;

(ii) $Cl(R[x]) = Cl(R) + U(R[x])$;

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Proposition (P. Kanwar, A. Leroy, J.M.)

Suppose that $Cl(R[x])$ *is a subring of* $R[x]$*. Then:*

(i) $Cl(R)$ *is a subring of* R*;*

(ii) $Cl(R[x]) = Cl(R) + U(R[x])$*;*

(iii) $R/N(R)$ *is a reduced ring.*

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Theorem (P. Kanwar, A. Leroy, J.M. (2015))

Let R be any ring. Then the following conditions are equivalent:

1. *The set $Cl(R[x])$ forms a subring of $R[x]$;*

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Proposition (P. Kanwar, A. Leroy, J.M.)

Suppose that $Cl(R[x])$ is a subring of $R[x]$. Then:

(i) *$Cl(R)$ is a subring of R;*

(ii) $Cl(R[x]) = Cl(R) + U(R[x])$;

(iii) *$R/N(R)$ is a reduced ring.*

Theorem (P. Kanwar, A. Leroy, J.M. (2015))

Let R be any ring. Then the following conditions are equivalent:

1. *The set $Cl(R[x])$ forms a subring of $R[x]$;*
2. *$Cl(R)$ is a subring of R and $Cl(R[x]) = Cl(R) + N(R)[x]$.*

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Proposition (P. Kanwar, A. Leroy, J.M. (2015))

Suppose R is 2-primal. Then $Cl(R[x])$ is a subring of $R[x]$ if and only if $Cl(R)$ is a subring of R.

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Theorem (P. Kanwar, A. Leroy, J.M.)

The following conditions are equivalent:

1. *The Köthe's problem has a positive solution;*
2. *For any clean ring R, the set $Cl(R[x])$ forms a subring of $R[x]$ if and only if $R/N(R)$ is a reduced ring;*

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Proposition (P. Kanwar, A. Leroy, J.M. (2015))

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Theorem (P. Kanwar, A. Leroy, J.M.)

The following conditions are equivalent:

1. *The Köthe's problem has a positive solution;*
2. *For any clean ring R, the set $Cl(R[x])$ forms a subring of $R[x]$ if and only if $R/N(R)$ is a reduced ring;*
3. *For any clean ring R such that the factor ring $R/N(R)$ is reduced, the set $Cl(R[x])$ is a subring of $R[x]$.*

Motivation

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Theorem (J.Han, , W. K. Nicholson (2001))

If R is a clean ring then so is the matrix ring $M_n(R)$.

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Theorem (J.Han, , W. K. Nicholson (2001))

If R is a clean ring then so is the matrix ring $M_n(R)$.

Question (A. J. Diesl (2013))

Let R be a nil clean ring. Is the matrix ring $M_n(R)$ nil clean, for any $n \in \mathbb{N}$?

Definitions

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Definition

- (A. J. Diesl) A nil clean ring R is uniquely nil clean if for every element a of R there exists unique idempotent $e \in R$ such that $a = e + n$, for some nilpotent $n \in R$.

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- (J.M.) A nil clean ring R is conjugate nil clean if for every element $a \in R$ and two nil clean decompositions $a = e + n = f + m$, the idempotents e and f are conjugate in R.

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Example

The ring $M_2(\mathbb{F}_2)$ is conjugate nil clean but it is not uniquely nil clean.

Result

Theorem (J.M. (2017))

Let us consider the following statements:

1. *For any nil clean ring R the matrix ring $M_n(R)$ is nil clean for all n;*

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4. *For any uniquely nil clean ring R the ring $M_2(R)$ is conjugate nil clean;*
5. *For any nil algebra A over $\mathbb{F}_2$ the matrix ring $M_n(A)$ is nil for all n;*

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4. *For any uniquely nil clean ring R the ring $M_2(R)$ is conjugate nil clean;*
5. *For any nil algebra A over $\mathbb{F}_2$ the matrix ring $M_n(A)$ is nil for all n;*
6. *Köthe's problem has positive solution in the class of $\mathbb{F}_2$-algebras.*

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Theorem (J.M. (2017))

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Then: • *statements* $(2), (3), (4), (5), (6)$ *are equivalent.*

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Theorem (J.M. (2017))

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6. *Köthe's problem has positive solution in the class of $\mathbb{F}_2$-algebras.*

Then: • *statements* (2), (3), (4), (5), (6) *are equivalent.*

• (1) *implies* (2).

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A ring R is said to be a *UJ*-ring if $1 + J(R) = U(R)$ (equivalently $U(R/J(R)) = \{1\}$).

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Proposition (M.T. Koşan, A. Leroy, J.M. (2018))

If the polynomial ring $R[x]$ is UJ, then R is a UJ-ring and $J(R)$ is a nil ideal of R.

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Definition

A ring R is said to be a *UJ*-ring if $1 + J(R) = U(R)$ (equivalently $U(R/J(R)) = \{1\}$).

Proposition (M.T. Koşan, A. Leroy, J.M. (2018))

If the polynomial ring $R[x]$ is UJ, then R is a UJ-ring and $J(R)$ is a nil ideal of R.

Proposition

Let R be a 2-primal UU-ring. Then, for any set X of commuting indeterminates, the polynomial ring $R[X]$ is a UJ-ring

Result

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Theorem (M.T. Koşan, A. Leroy, J.M. (2018))

The following conditions are equivalent:

1. *For any UJ-ring R with nil Jacobson radical, the polynomial ring $R[x]$ is also UJ;*

Result

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Theorem (M.T. Koşan, A. Leroy, J.M. (2018))

The following conditions are equivalent:

1. *For any UJ-ring R with nil Jacobson radical, the polynomial ring $R[x]$ is also UJ;*
2. *Köthe's problem has a positive solution in the class of algebras over $\mathbb{F}_2$.*

Thanks

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Thanks for your attention