The aim of this poster is to give a short introduction to
1. Hopf-Galois Structures
2. Skew Braces and 3, 4 their Relationship
5. Skew Braces and 7 Hopf-Galois Structures Classification
6. Automorphism Groups of Skew Braces and Examples
8. Skew Braces of Semi-direct Product Type

For simplicity we assume $L/K$ is a Galois extension of fields with Galois group $G$.

1. Hopf-Galois Structures

A Hopf-Galois structure on $L/K$ consists of a finite dimensional cocommutative $K$-Hopf algebra $H$ together with an action on $L$ which makes $L$ into an $H$-Galois extension.

2. Skew Braces

a (left) skew brace is a triple $(B, \oplus, \circ)$ which consists of a set $B$ together with two operations $\oplus$ and $\circ$ such that $(B, \oplus)$ and $(B, \circ)$ are groups (neither necessarily abelian), and the two operations are related by the skew brace property:

$$a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$$
for every $a, b, c \in B$, where $\oplus$ is the inverse of $a$ with respect to the operation $\oplus$. Braces were introduced by Rump in 2007. Many properties of braces were investigated by Bachiller, Cedó, Jespers, Okniński et al.

Skew braces, as a generalisation of braces, and their connections to other areas, were studied by Byott, Guarrini, Smoktunowicz, and Vendramin.

Notation: We call a $G$-skew brace of type $N$ a skew brace $(B, \oplus, \circ)$ such that $(B, \oplus) \cong N$ and $(B, \circ) \cong G$.

Skew Braces of Order $p^3$ for $p \geq 3$
The number of $G$-skew braces of type $N$, $e(G, N)$, is given by

<table>
<thead>
<tr>
<th>$e(G, N)$</th>
<th>$C_p$</th>
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Note $e(G, N) = e(N, G)$.

3. From Hopf-Galois Structures to Skew Braces

- Suppose $H$ endows $L/K$ with a Hopf-Galois structure.
- Then $H = L[N]^G$ for some $N \subseteq \text{Perm}(G)$ which is a regular subgroup normalised by the left translations.
- The subgroup $N$ is a regular implies that we have a bijection $\phi : N \to G$.
- $n \mapsto n \cdot 1_G$.
- Set $(B, \oplus) = N$ and define a new group operation by $n_1 \circ n_2 = \phi^{-1}(\phi(n_1) \phi(n_2))$ for $n_1, n_2 \in N$.
- The subgroup $N$ is normalised by the left translations implies that $(B, \oplus, \circ)$ is a $G$-skew brace of type $N$ corresponding to $H$.

4. From Skew Braces to Hopf-Galois Structures

- Suppose $(B, \oplus, \circ)$ is a $G$-skew brace of type $N$.
- The map $d : (B, \oplus) \to \text{Perm}(B, \circ)$ defined by $a \to (da : b \to a \circ b)$ is a regular embedding.
- The skew brace property implies that for all $a, b, c \in B$ $b \circ (da \circ b^{-1} c) = d(a \circ b^{-1}) c$, i.e., $bd_b b^{-1} = d(b \circ a) b^{-1}$.
- Thus $L[(B, \oplus)]^{(B, \circ)}$ endows $L/K$ with a Hopf-Galois structure corresponding to the skew brace $(B, \oplus, \circ)$.

Problem

The group Perm$(G)$ can be large.

5. Classifying Skew Braces: working with holomorphs

For a skew brace $(B, \oplus, \circ)$ the map $m : (B, \oplus) \to \text{Hol}(B, \oplus)$ defined by $a \to (ma : b \to a \circ b)$ is a regular embedding, where $\text{Hol}(B, \oplus) = (B, \oplus) \rtimes \text{Aut}(B, \circ)$. For $f : (B, \oplus, \circ_1) \to (B, \oplus, \circ_2)$ an isomorphism of skew braces, we have $(B, \circ_1) \cong (B, \circ_2)$ and $f$ is the conjugation by $f$.

Classifying Skew Braces

To find the non-isomorphic $G$-skew braces of type $N$ for a fixed $N$, classify elements of the set $\{H \subseteq \text{Hol}(N) \mid H \text{ is regular, } H \cong G\}$, and extract a maximal subset whose elements are not conjugate by any element of Aut$(N)$.

6. Upshot: Automorphism Groups of Skew Braces

We find $\text{Aut}_{B_0}(B, \oplus, \circ) \cong \{a \in \text{Aut}(B, \oplus) \mid a | \text{Im} m \alpha \cong \text{Im} m \}$.

Example (Skew Braces of $C_p = \langle \sigma \mid \sigma^{p+1} = 1 \rangle$ type for $p \geq 3$ and $n > 1$)

$\text{Hol}(C_p) = \langle \sigma \mid \text{Im} m \rangle$ with $\text{Aut}_{B_0}(C_p)$ is given by $\langle \sigma^m \rangle$.

7. Finding Hopf-Galois Structures

Denote by $B_0^G$ the isomorphism class of a $G$-skew brace of type $N$ given by $(B, \oplus, \circ)$. Then the number of Hopf-Galois structures on $L/K$ of type $N$ is given by

$$\sum_{B_0^G} \left| \frac{\text{Aut}(G)}{\text{Aut}_{B_0}(B_0^G)} \right|$$

Hopf-Galois Structures of Order $p^3$ for $p > 3$
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