Indecomposable left cycle sets

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Definition

Let X be a non-empty set, $r : X \times X \to X \times X$ a map and write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

for all $x, y \in X$.
Then (X, r) is said to be a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation if and only if the following properties hold:

(1) $r^2 = id_{X^2}$; (r is involutive)

(2) $\lambda_x, \rho_x \in Sym_X$ for all $x \in X$; (r is non degenerate)

(3) $r_1 r_2 r_1 = r_2 r_1 r_2$

where $r_1 = r \times id_X$ and $r_2 = id_X \times r$.

Convention: From now on, by a solution I mean a non-degenerate involutive set-theoretic solution.

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In the study of all the solutions, the indecomposable solutions play a key-role.

Definition (P. Etingof, T.Schedler, A.Soloviev, 1999)

A solution (X, r) is said to be *decomposable* if there exists a non trivial partition $X = Y \cup Z$ with $r(Y \times Y) \subseteq Y \times Y$ and $r(Z \times Z) \subseteq Z \times Z$, such that the restrictions of r to $Y \times Y$ and $Z \times Z$ are again non-degenerate.
Otherwise it is said *indecomposable*.

Definition (Rump, 2005)

A pair $(X, \cdot)$ is said a *non-degenerate left cycle set* if X is a non-empty set, and $\cdot$ a binary operation on X such that

1) $(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z)$ for all $x, y, z \in X$;
2) the left multiplication $\sigma_x : X \longrightarrow X,\ y \longmapsto x\cdot y$ is bijective for every $x \in X$;
3) $\mathfrak{q} : X \longrightarrow X,\ x \longmapsto x\cdot x$ is bijective.

We will call $(X, \cdot)$ *square-free* if $\mathfrak{q} = id_X$.

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Theorem (Rump, 2005)

If (X, r) is a solution, where $r(x, y) := (\lambda_x(y), \rho_y(x))$ then $(X, \cdot)$ is its **left cycle set associated** *defined by*

$$\sigma_x := \lambda_x^{-1}$$

for every $x \in X$.

Vice versa if $(X, \cdot)$ is a left non-degenerate cycle set and σ_x its left multiplication then, for all $x, y \in X$

$$r(x, y) := (\sigma_x^{-1}(y), \sigma_x^{-1}(y) \cdot x)$$

is its **associated solution**.

Furthermore, a solution (X, r) is indecomposable if and only if the associated left cycle set is indecomposable.

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Let X be a non-degenerate left cycle set, and $\sigma : X \to Sym_X$, $x \mapsto \sigma_x$. We denote by $\mathcal{G}(X)$ the subgroup of Sym_X generated by the image $\sigma(X)$ of σ and we call it the **associated permutation group**.

Example

Let X be the cycle set given by

$\cdot$	1	2	3	4
1	2	1	4	3
2	4	3	2	1
3	2	1	4	3
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Then X is an indecomposable left cycle set and the group $\mathcal{G}(X)$ is the Klein group.

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Theorem (Rump, 2005)

Every indecomposable finite left cycle set is not square-free.

Recall that a non-degenerate left cycle set is *square-free* if $\mathfrak{q}: X \to X$, $x \mapsto x \cdot x$ is such that $\mathfrak{q} = id_X$.

Theorem (P. Etingof, T.Schedler, A.Soloviev, 1999)

Let X be an indecomposable left cycle set of cardinality a prime number p. Then X is isomorphic to the left cycle set $(\mathbb{Z}/p\mathbb{Z}, \cdot)$ given by $x \cdot y := y + 1$ for every $x, y \in \mathbb{Z}/p\mathbb{Z}$.

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Proposition (P. Etingof, T.Schedler, A.Soloviev, 1999)

A non-degenerate left cycle set X is indecomposable if and only if the associated permutation group $\mathcal{G}(X)$ is transitive on X.

Example

Let $X := \{1,2,3,4,5,6,7,8\}$ be the left cycle set given by

$$\sigma_1 := (46) \quad \sigma_2 := (35)$$

$$\sigma_3 := (28) \quad \sigma_4 := (17)$$

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Definition (Vendramin, 2015)

Let X be a left cycle set, S a non-empty set and $\alpha : X \times X \times S \longrightarrow Sym(S)$, $(i, j, s) \mapsto \alpha_{i,j}(s, -)$. Then α is said **dynamical cocycle** of X if and only if

$$\alpha_{i \cdot j, i \cdot k}(\alpha_{i,j}(r, s), \alpha_{i,k}(r, t)) = \alpha_{j \cdot i, j \cdot k}(\alpha_{j,i}(s, r), \alpha_{j,k}(s, t)).$$

for every $i, j, k \in X$, $s, t \in S$.

Proposition (Vendramin, 2015)

If α is a dynamical cocycle then $S \times_\alpha X := (S \times X, \cdot)$ is a left cycle set, where

$$(s, i) \cdot (t, j) := (\alpha_{i,j}(s, t), i \cdot j),$$

and we will call $S \times_\alpha X$ **dynamical extension** *of X by α. Moreover, $S \times_\alpha X$ is said* **trivial** *if $|X| = 1$ or $|S| = 1$.*

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Let $S \times_\alpha X$ be a dynamical extension of X by α.

Proposition (M. C., F. Catino, G. Pinto, submitted)

For every $i \in X$ let H_i be the subgroup of $\mathcal{G}(S \times_\alpha X)$ given by

$$H_i := \{h \in \mathcal{G}(S \times_\alpha X) \quad | \quad \forall s \in S \quad h(s,i) \in S \times \{i\}\}.$$

Then $S \times_\alpha X$ is indecomposable if and only if X is indecomposable and H_i is transitive on $S \times \{i\}$ for some $i \in X$.

A large family of dynamical extensions is obtained by M. C., Francesco Catino and Giuseppina Pinto.
Let I be a left non-degenerate cycle set, A, B non-empty sets, $\gamma : B \longrightarrow Sym(A)$, $\beta : A \times A \times I \longrightarrow Sym(B)$ and $S := A \times B$.
Let $\alpha : I \times I \times S \longrightarrow Sym(S)$ be the function given by

$$\alpha_{i,j}((a,b),(c,d)) := \begin{cases} (c, \beta_{(a,c,i)}(d)), & \text{if } i = j \\ (\gamma_b(c), d), & \text{if } \ i \neq j \end{cases}.$$

for all $(i, j, (a, b)) \in I \times I \times S$ and $(c, d) \in S$.

Theorem (M. C., F. Catino, G. Pinto, 2017)

If $\gamma : B \longrightarrow Sym(A)$ *and* $\beta : A \times A \times I \longrightarrow Sym(B)$ *are such that*

1) $\gamma_b \gamma_d = \gamma_d \gamma_b$,
2) $\beta_{(a,c,i)} = \beta_{(\gamma_b(a), \gamma_b(c), j \cdot i)}$,
3) $\gamma_{\beta_{(a,c,i)}(d)} \gamma_b = \gamma_{\beta_{(c,a,i)}(b)} \gamma_d$,
4) $\beta_{(a,c,i \cdot i)} \beta_{(a',c,i)} = \beta_{(a',c,i \cdot i)} \beta_{(a,c,i)}$

hold for all $a, a', c \in A$, $b, d \in B$ *and* $i, j \in I$, $i \neq j$, *then* α *is a dynamical cocycle and hence* $S \times_\alpha I$ *is a non-degenerate left cycle set.*

Example

Let $A = B := \mathbb{Z}/2\mathbb{Z}$, $I := \{1, 2\}$ the left cycle set given by $\sigma_1 = \sigma_2 := (12)$, $\gamma : B \longrightarrow Sym(A)$ and $\beta : A \times A \times I \longrightarrow Sym(B)$ given by

$$\gamma_c := t_{-c-1} \qquad \beta_{(a,a,i)} := id_B \qquad \beta_{(a,b,i)} := (1 \quad 2)$$

for all $a, b \in A$, $a \neq b$, $c \in B$, $i \in I$, where $t_c(x) := x + c$ for all $x, c \in \mathbb{Z}/2\mathbb{Z}$.
Then $(A \times B) \times_\alpha I$ is indecomposable: indeed H_1 is isomorphic to $Sym(A \times B \times \{1\}) \cong Sym(4)$ and I is indecomposable.

Example

Let I be an indecomposable left cycle set and $A = B := \mathbb{Z}/k\mathbb{Z}$. Put

$$\beta_{(a,a,i)} := id_A \qquad \beta_{(a,b,i)} := t_1 \qquad \gamma_a := t_{-a-1}$$

for all $i \in I$ and $a \in A$, $b \in B$, $a \neq b$ where $t_a(x) := x + a$ for all $x, a \in \mathbb{Z}/k\mathbb{Z}$.
Then $(A \times B) \times_\alpha I$ is an indecomposable left cycle set of cardinality $|I|k^2$.

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THANKS FOR YOUR ATTENTION!