Contraction Algebras and their Properties

Michael Wemyss

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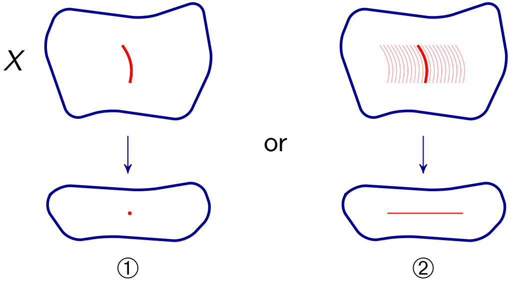
The Geometric Setup

Consider C, a single contractible curve in a smooth CY 3-fold X. In cartoons, this means



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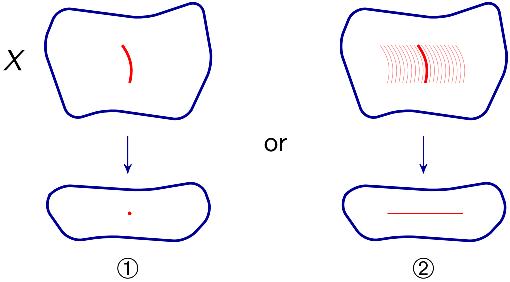


The basic idea of this talk:

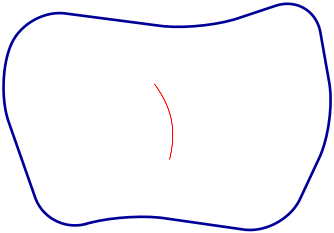
$$C \text{ in } X \;\overset{\text{associate}}{\rightsquigarrow}\; \text{an algebra } A_{\mathrm{con}}$$

How to do this?

There are four ways of constructing this algebra.

1. Deformation Theory

Probing how the curve *deforms* is one way to obtain good information about its behaviour.



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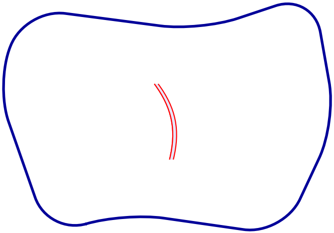

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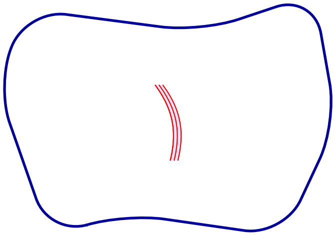

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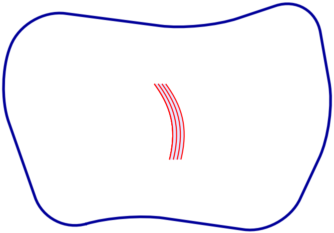

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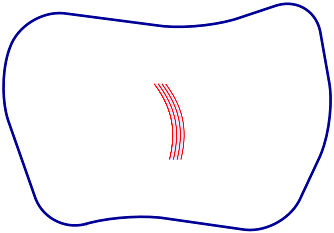

Noncommutative Deformation Theory (Laudal, Segal, ELO): there is a functor, giving rise to a noncommutative algebra...

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Via various isomorphisms (Donovan–W), it is possible to view A_{con} in the following, explicit, form.

4. Superpotential Algebras

There exists an $f \in \mathbb{C}\langle x, y\rangle$ such that

$$A_{\text{con}} \cong \frac{\mathbb{C}\langle x, y\rangle}{(\delta_x f, \delta_y f)} = J_f$$

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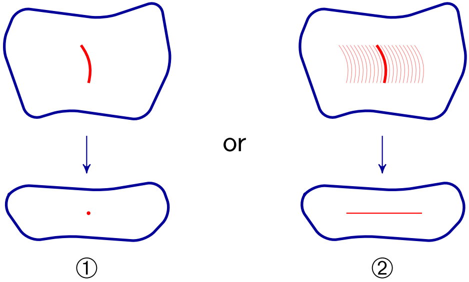
Calibration: if $f = x^4 + xyy + yxy + yyx$, then

$$\delta_x f = x^3 + y^2 \quad \text{and} \quad \delta_y f = xy + yx.$$

The Contraction Theorem

Recall our setup:



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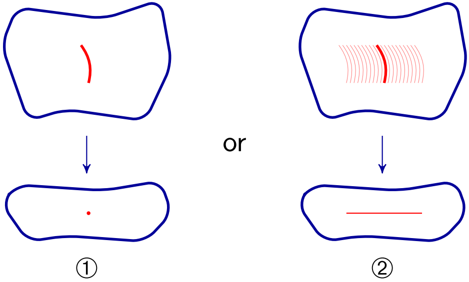

Theorem (Donovan–W)

1. Situation ① $\iff$ $\mathrm{A}_{\mathrm{con}}$ is finite dimensional.
2. $\mathrm{A}_{\mathrm{con}}$ controls the symmetries, in both situations.

The Two Main Conjectures

Rest of talk: situation ① (i.e. flopping contractions).

The Classification Problem (Donovan–W)

Let $X \to \operatorname{Spec} R$ and $Y \to \operatorname{Spec} S$ be two 3-fold flops, with associated contraction algebras A_{con} and B_{con}. Then

$$X \sim Y \iff A_{\text{con}} \cong B_{\text{con}}.$$

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The Realisation Problem (Brown–W)

Every finite dimensional superpotential algebra

$$J_f = \frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{(\delta_x f, \delta_y f)}$$

can be constructed as the contraction algebra of some 3-fold flop.

Strange Behaviour 1

First, consider the following six algebras:

$$\mathbb{C}, \quad \frac{\mathbb{C}\langle x,y,z\rangle.}{\substack{x+y+z=0\\ x^2=0\\ y^2=0\\ z^2=0}}, \quad \frac{\mathbb{C}\langle x,y,z\rangle}{\substack{x+y+z=0\\ x^2=0\\ y^3=0\\ z^3=0}}, \quad \frac{\mathbb{C}\langle x,y,z\rangle}{\substack{x+y+z=0\\ x^2=0\\ y^3=0\\ z^4=0}}$$

$$\frac{\mathbb{C}\langle x,y,z\rangle}{\substack{x+y+z^2=0\\ \text{messy}}}, \quad \frac{\mathbb{C}\langle x,y,z\rangle}{\substack{x+y+z=0\\ x^2=0\\ y^3=0\\ z^5=0}}.$$

These have dimensions 1, 4, 12, 24, 40 and 60 respectively.

Now, consider the centre of $J_f \cong A_{\mathrm{con}}$, with basis

$$\{1 = c_1, c_2, \dots, c_n\}.$$

Consider a generic central element $s = \sum_i \lambda_i c_i$, which means that (λ_i) belongs to a Zariski open subset of $\mathbb{A}^n$.

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Theorem (Donovan–W)

$A_{\text{con}}/(s)$ is isomorphic to one of the six algebras on the last slide.

Label the cases $\ell = 1, \ldots, 6$ (where $\ell = 1$ corresponds to the algebra of dimension one, and $\ell = 6$ the algebra of dimension 60).

Strange Behaviour 2

Theorem (Hua–Toda)

There is an equality

$$\dim_{\mathbb{C}} A_{\text{con}} = \underbrace{\dim_{\mathbb{C}} A_{\text{con}}^{ab}}_{n_1} + \sum_{i=2}^{\ell} n_i \cdot i^2,$$

where ℓ is determined by the last slide, such that *all* $n_i \neq 0$.

The n_i are called the Gopakumar–Vafa (GV) invariants.

The GV invariants are a property of the isomorphism class of A_{con}, but it is still not known how to extract them intrinsically.

Upshot

Given $f \in \mathbb{C}\langle\langle x, y \rangle\rangle$ with $\dim_{\mathbb{C}} J_f < \infty$, the conjectures (and numerical evidence!) *predict* the following algebraic statements:

- A generic central cut $J_f/(s)$ is one of six algebras, so there is an ADE-type classification of such J_f.
- The dimension of J_f is a sum of squares,

  $$\dim_{\mathbb{C}} J_f = \dim_{\mathbb{C}} J_f^{ab} + n_2 \cdot 2^2 + \ldots + n_\ell \cdot \ell^2$$

  with all $n_i \neq 0$.
- J_f is a symmetric algebra ($\mathrm{Hom}_{\mathbb{C}}(J_f, \mathbb{C}) \cong J_f$ as bimodules).

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- J_f is a symmetric algebra ($\mathrm{Hom}_{\mathbb{C}}(J_f, \mathbb{C}) \cong J_f$ as bimodules).
- Furthermore, 3-fold flops are classified by certain elements in the free algebra in two variables.