Lie solvability in matrix algebras

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Noncommutative and non-associative structures, braces and applications
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Based on a joint works with J. van den Berg, J. Szigeti and L. van Wyk

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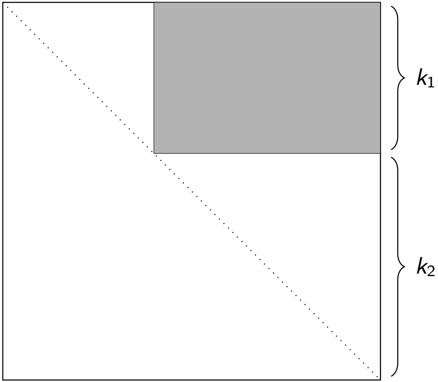
$J =$ 

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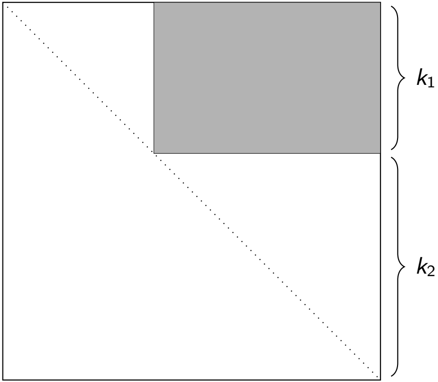
$J =$ 

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- Commutativity:

$$\forall r, s \in R,\ [r, s] \stackrel{\text{def}}{=} rs - sr.$$

Define inductively the Lie central and Lie derived series of a ring R as follows:

$$\mathfrak{C}^0(R) := R, \ \mathfrak{C}^{q+1}(R) := [\mathfrak{C}^q(R), R] \text{ (central series)}, \tag{1}$$

and

$$\mathfrak{D}^0(R) := R, \ \mathfrak{D}^{q+1}(R) := [\mathfrak{D}^q(R), \mathfrak{D}^q(R)] \text{ (derived series)}. \tag{2}$$

We say that R is Lie nilpotent (respectively, Lie solvable) of index q (for short, R is Ln_q; respectively, R is Ls_q) if $\mathfrak{C}^q(R) = 0$ (respectively, $\mathfrak{D}^q(R) = 0$).

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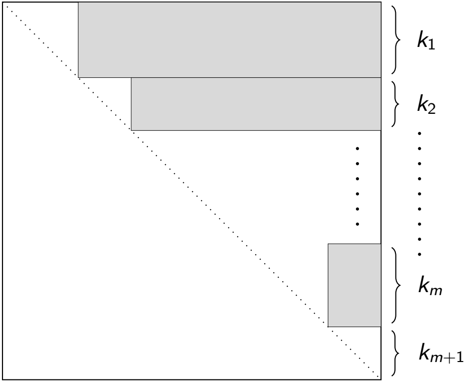
$J =$ 

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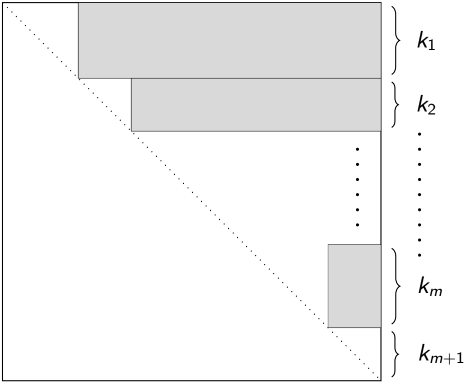

- Let $R = FI_n + J$ (“TYPICAL EXAMPLE”)

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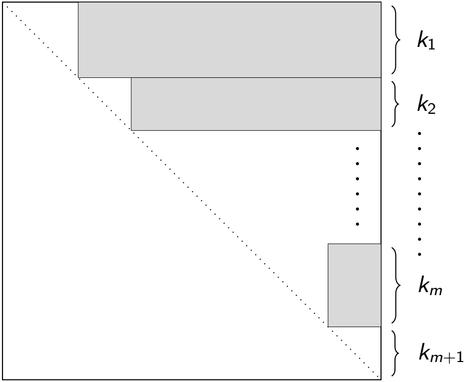

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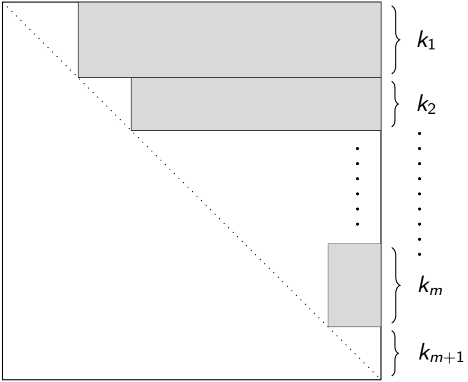

- $$\dim_F R = k_1(n-k_1) + k_2(n-k_1-k_2) + \cdots + k_m(n-k_1-k_2-\cdots-k_m) + 1 = \sum_{i,j=1,\, i<j}^{m+1} k_i k_j + 1.$$

- $$M(\ell, n) \stackrel{\text{def}}{=} \max \left\{ \sum_{i,j=1,\, i<j}^{\ell} k_i k_j + 1 : k_1, k_2, \ldots, k_\ell \text{ are nonnegative integers such that } \sum_{i=1}^{\ell} k_i = n \right\}.$$

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- Let $\ell \leqslant n$ and
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- We get $M(\ell, n)$ for the sequence $(k_1, k_2, \ldots, k_\ell) \in \mathbb{N}_0^\ell$ defined in the following way:

$$k_i \stackrel{\text{def}}{=} \begin{cases} \left\lfloor \frac{n}{\ell} \right\rfloor, & \text{for } 1 \leqslant i \leqslant \ell - r \\ \left\lfloor \frac{n}{\ell} \right\rfloor + 1, & \text{for } \ell - r < i \leqslant \ell. \end{cases}$$

- **Conjecture.** *(J. Szigeti, L. van Wyk)* Let F be any field, m and n positive integers, and R an F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m. Then

$$\dim_F R \leqslant M(m+1, n).$$

Theorem 1

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- Unfortunately, we do not have good "typical example".

FACTS:

- $$[x_1, y_1]\,[x_2, y_2] \cdots [x_q, y_q] = 0 \tag{3}$$

  Mal'tsev proved that all the polynomial identities of $U_q(F)$ are consequences of the identity in (3).

  We denote algebras satisfying (3) by D_{2^q}.

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- $$\mathcal{D} = \left\{ \begin{bmatrix} D_1 & D_{(1,2)} & \cdots & D_{(1,q)} \\ 0 & D_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & D_{(q-1,q)} \\ 0 & \cdots & 0 & D_q \end{bmatrix} \right\} \qquad (4)$$

  Each D_i is a commutative F-subalgebra of $M_{n_i}(F)$ for every i, and $D_{(j,k)} = M_{n_j \times n_k}(F)$ for all j and k such that $1 \le j < k \le q$. $\mathcal{D}$ satisfies (3).

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- (Meyer, Szigeti, van Wyk) For any commutative ring R, the subring $U_3^\star(U_3^\star(R))$ of $U_9^\star(R)$ is Ls_2, but it is neither Ln_2 nor D_2, and so we have, in general,

$$\mathrm{Ls}_2 \quad \not\Rightarrow \quad \mathrm{Ln}_2 \text{ or } \mathrm{D}_2. \tag{7}$$

Problem 2

Construct an example of Ls_2 subalgebra of $M_n(F)$ with dimension bigger than $2 + \left\lfloor \frac{3n^2}{8} \right\rfloor$.

Theorem 3

If A is an Ls_{m+1} (for some $m \geq 1$) structural matrix subring of $U_n(R)$, R a commutative ring and $n \geq 1$, then A is D_{2^m}.

Let k be a positive integer and $n = 2k + 1$. Consider

$$A = \left\{ \left(\begin{array}{c|c} A_1 & B \\ \hline 0 & A_2 \end{array} \right) : A_1 \in M_k(F), A_2 \in M_{k+1}(F), A_i - \text{comm.} \right\}.$$

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Theorem 4

If A is a D_2 subalgebra of $U_n(F)$ with maximum possible dimension for D_2, such that A_1, A_2 and B are independent, then A_1 and A_2 are commutative.

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Thank you for your attention!