Simply connected latin quandles

Petr Vojtěchovský

University of Denver

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Coauthors

This is joint work with Marco Bonatto.

Outline

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Racks and quandles

Definition

A **rack** is a groupoid $(Q, \cdot)$ in which

- all left translations $L_x : Q \to Q$, $y \mapsto xy$ are bijections of Q,
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Quandles can be used to consistently color arcs of oriented knot diagrams.

Enumeration of racks and quandles - as of two weeks ago

Nelson and McCarron enumerated racks r_n and quandles q_n of order n up to isomorphism for small values of n:

n	1	2	3	4	5	6	7	8	9	10
r_n	1	2	6	19	74	353	2080	16023	?	?
q_n	1	1	3	7	22	73	298	1581	11079	?

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Theorem (Blackburn 2013, Ashford + Riordan 2017)

Let $\varepsilon > 0$. Then for all sufficiently large n the number $\sharp$ of racks of order n up to isomorphism satisfies

$$2^{(1/4-\varepsilon)n^2} \leq \sharp \leq 2^{(1/4+\varepsilon)n^2}.$$

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The moral is: There are very many racks and quandles already for small orders. Specialize!

Connected quandles

Definition

For a rack Q let $\mathrm{Mlt}_\ell(Q) = \langle L_x : x \in Q\rangle$ be the **left multiplication group** of Q. (This is also denoted in rack literature by $\mathrm{Inn}(Q)$ and called the **inner automorphism group**.)

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A quandle Q is **connected** (or **transitive**) if $\mathrm{Mlt}_\ell(Q)$ acts transitively on Q. It is **doubly transitive** if $\mathrm{Mlt}_\ell(Q)$ acts doubly transitively on Q.

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Definition

A quandle Q is **latin** if every right translation $R_x : Q \to Q$, $y \mapsto yx$ is a bijection of Q.

Latin quandles are connected.

Affine quandles

Definition

Let $(G,+)$ be an abelian group and $\alpha\in\mathrm{Aut}(G,+)$. Then $X=\mathcal{Q}(G,\alpha)$ defined by

$$x*y=x+\alpha(y-x)$$

is an **affine quandle**.

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Let X be a groupoid and S a set. The canonical projection $X \times S \to X$ is a homomorphism if and only if

$$(x,s)(y,t) = (xy, \beta(x,y,s,t))$$

for some $\beta : X \times X \times S \times S \to S$. Call it $X \times_\beta S$.

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Moreover, $X \times_\beta S$ is a quandle if and only if $\beta(x, y, s) : S \to S$, $t \mapsto \beta(x, y, s, t)$ is always a bijection and the following conditions hold:

$$\beta(xy, xz, \beta(x, y, s)(t))\beta(x, z, s) = \beta(x, yz, s)\beta(y, z, t),$$
$$\beta(x, x, s)(s) = s.$$

We call β a (dynamical) **quandle cocycle**.

Coverings and constant quandle cocycles

If $\beta(x, y, s) = \beta(x, y, t)$ for all x, y, s, t, we think of β as a mapping $X \times X \to \mathrm{Sym}(S)$ and we call it a **constant quandle cocycle**.

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Definition (Eisermann)

A connected quandle Y is a **covering** of a quandle X if there is a surjective quandle homomorphism $f: Y\to X$ such that $L_x=L_y$ holds in Y whenever $f(x)=f(y)$.

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Proposition

The following conditions are equivalent for connected quandles X, Y:

- Y *is a covering of* X,
- Y *is isomorphic to* $X \times_\beta S$ *for some set* S *and some constant quandle cocycle* β.

Cohomologous cocycles

Definition (A+G)

Two quandle cocycles β, $\beta' : X \times X \times S \to \mathrm{Sym}(S)$ are **cohomologous** (write $\beta \sim \beta'$) if there is $\gamma : X \to \mathrm{Sym}(S)$ such that

$$\beta'(x, y, s) = \gamma(xy)\beta(x, y, \gamma(x)^{-1}(s))\gamma(y)^{-1}.$$

Let $H^2_c(X, \mathrm{Sym}(S))$ be the **second cohomology group** of constant quandle cocycles with coefficients in $\mathrm{Sym}(S)$.

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Definition (E)

A quandle Q is **simply connected** if it is connected and $H^2_c(X, \mathrm{Sym}(S)) = 1$ for every set S. (The second condition says that every covering of X is equivalent to the trivial covering of X over some set S.)

Main result

Our goal is to exhibit classes of simply connected latin quandles. We develop a combinatorial approach to constant quandle cocycles and prove:

Theorem (MB+PV)

Let X be a finite connected quandle that is either

- *affine over a cyclic group, or*
- *doubly transitive of order different from 4.*

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- The first part is easy and can be proved in a number of ways.
- The second part is quite delicate when the combinatorial approach is used. There is also a higher-level proof (still somewhat detailed) based on an unpublished observation of Clauwens about the fundamental groups of quandles.

Normalized constant cocycles

Definition

Let X be a connected quandle and $u \in X$ be fixed. A constant quandle cocycle β is u-**normalized** if $\beta(x, u) = 1$ for every $x \in X$.

In a latin quandle, every constant quandle cocycle is cohomologous to a u-normalized cocycle.

Main idea of the combinatorial approach

- Think of a constant quandle cocycle $\beta : X \times X \to \mathrm{Sym}(S)$ as a square array with entries in $\mathrm{Sym}(S)$.

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- Study the orbits of these bijections.
- To deduce that X is simply connected, show that the orbits union to all of $X \times X$ for every S and β.

Bijections that preserve cocycle values

The following bijections were used in the proof:

$$\begin{aligned} f(x,y) &= (x(y/u), xu), \\ g(x,y) &= (xu, yu), \\ h(x,y) &= ((y/(x\backslash u))x, y), \end{aligned}$$

and we looked at the orbits of the group $G = \langle f, g, h \rangle$ on $X \times X$.

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- There is a doubly transitive quandle of order 4 that is not simply connected.

The fundamental group of a quandle

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Let X be a quandle and $x \in X$. The **fundamental group** of X at x is the group

$$\pi_1(X, x) = \{g \in \mathrm{Ker}(\varphi) : x^g = x\}.$$

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Proposition

Let X be a connected quandle. Then X is simply connected iff $\pi(X) = 1$.

Clauwens' characterization of $\pi(X)$ for affine quandles

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Theorem (Clauwens, unpublished)

Let $X = \mathcal{Q}(G, \alpha)$ be an affine quandle. Then $\pi_1(X) \cong S(X)$.

Finishing the proof

Proposition (Vendramin + ε)

Let X be a finite quandle. Then X is doubly transitive iff $X \cong \mathcal{Q}(\mathbb{Z}_p^n, \alpha)$ for some prime p, integer n and $\alpha \in \mathrm{Aut}(\mathbb{Z}_p^n)$ with $|\alpha| = |X| - 1$.

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It now suffices to show $I(X) = G \otimes G$. About 2 pages of calculations.