Algebras birational to generic Sklyanin algebras

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Throughout, k is an algebraically closed field.

Definition

Let $a, b, c \in k$. The (3-dimensional) Sklyanin algebra is

$$\begin{aligned} S = S_{abc} = k\langle x, y, z\rangle / axy + byx + cz^2, \\ ayz + bzy + cx^2, \\ azx + bxz + cy^2. \end{aligned}$$

If $[a : b : c] \in \mathbb{P}^2 \smallsetminus \{\text{ 12 points }\}$ then S has Hilbert series $1/(1-t)^3$. (We'll always assume this.)

If $[a : b : c]$ is generic enough, S is very noncommutative: $Z(S) = k[g]$ where $g \in S_3$. Again, we will assume this.

The Skylanin relations on S_1 satisfy the YBE:

$$\begin{tikzcd}[column sep=small]
 & -\frac{a}{b}yzx - \frac{c}{b}x^3 \arrow[r, "R_{23}"] & yxz + \frac{c}{b}(-x^3+y^3) \arrow[dr, "R_{12}"] & \\
zyx \arrow[ur, "R_{12}"] \arrow[dr, "R_{23}"'] & & & -\frac{a}{b}xyz + \frac{a}{b}(-x^3+y^3-z^3) \\
 & -\frac{a}{b}zxy - \frac{c}{b}z^3 \arrow[r, "R_{12}"'] & xzy + \frac{c}{b}(y^3-z^3) \arrow[ur, "R_{23}"'] &
\end{tikzcd}$$

S is 3-CY, potential algebra, Artin-Schelter regular, of I-type, and (Artin-Tate-Van den Bergh) is a noetherian domain.

We think of S as the coordinate ring of $\mathbb{P}^2_{NC}$, in the same way that $S_{1,-1,0} = k[x,y,z]$ is the coordinate ring of $\mathbb{P}^2$.

We report on an ongoing joint project with Dan Rogalski and Toby Stafford.

Goal: classify connected graded (left and right) noetherian R which are orders in

$$Q_{gr}(S) := S\langle h^{-1} : h \in S^* \text{ homogeneous } \rangle.$$

We say such R are birational to S.

(Reminder: R is connected graded (cg) if $R = \bigoplus_{n \geq 0} R_n$ is $\mathbb{N}$-graded with $R_0 = k$.)

The goal is part of Artin's programme to classify NC graded domains of GK-dimension 3.

For technical reasons, we first study

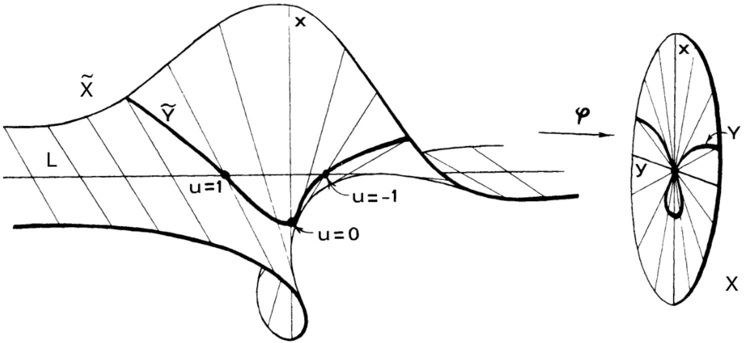
$$T := S^{(3)} = \bigoplus_{n \geq 0} S_{3n}$$

and cg noetherian algebras birational to T.

We will see that there is a beautiful analogy with the algebraic geometry of rational projective surfaces (surfaces birational to $\mathbb{P}^2$). We obtain results that mirror the commutative results extremely precisely.

These results have powerful unexpected consequences for the classification project.

Blowing up a point on a commutative surface: replace $p = (0,0) \in X$ by a line L to get $\widetilde{X} = \mathrm{Bl}_p(X)$.



Picture due to R. Hartshorne

- φ is isomorphism away from p.
    - $D^b(\mathrm{coh}\,\widetilde{X}) \simeq D^b(\mathrm{coh}\,X) \oplus \mathrm{add}(L)$ (Orlov)
- $L = \varphi^{-1}(p) \cong \mathbb{P}^1$ (the exceptional line)
- $\mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L) = 0$.

We can also blow up a point on T (or S).

But first: What is a point?

Let $V \in \mathbb{P}(S_1^*)$ – i.e. $V \subset S_1$, $\dim V = 2$.

Fact: There is a smooth cubic curve $E \subset \mathbb{P}(S_1^*)$ so that

$$V \in E \iff \dim S/VS = \infty.$$

In this case, $\dim S_n/VS_{n-1} = 1$ for all n: S/VS is a <u>point module</u>

Definition

Let $V \in E$. Define

$$R = \mathrm{Bl}_V(T) = k\langle VS_2\rangle \subset T.$$

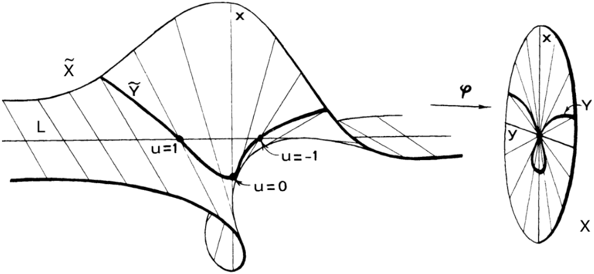
Theorem (Rogalski 2009)

Let $R = \mathrm{Bl}_V(T)$ as above. Then R is cg noetherian and birational to T.

Further, there is a module L_R so that $(T/R)_R \cong \bigoplus_{n\geq 1} L(-n)$.

- hilb $L = 1/(1-t)^2$, *that is L is a* <u>*line module*</u>
- $\mathrm{Ext}^1_R(L, L) = 0$ *(Rogalski-S.-Stafford)*
- $D^b(\text{qgr-}R) \simeq D^b(\text{qgr-}T) \oplus \text{add}\, L$ *(Van den Bergh)*

We are building an analogy between geometry and NC algebra:



geometry	algebra
$\mathbb{P}^2$	T
$p \in \mathbb{P}^2$	$V \in E$
$\varphi^{-1} : \mathbb{P}^2 \dashrightarrow \mathrm{Bl}_p(\mathbb{P}^2)$	$T \supset \mathrm{Bl}_V(T)$
exceptional line L	line module L
$\mathrm{Ext}^1(\mathcal{O}_L, \mathcal{O}_L) = 0$	$\mathrm{Ext}^1(L, L) = 0$

Theorem

(Rogalski 2009) Any cg noetherian subalgebra of T that is a maximal order in $Q_{\mathrm{gr}}(T)$ and generated in degree 1 is equal to an iterated blowup of T at ≤ 7 points.

(Rogalski-S.-Stafford 2013) Any cg noetherian subalgebra of T that is an order in $Q_{\mathrm{gr}}(T)$ is an equivalent order to an iterated blowup of T at ≤ 8 points. We classify subalgebras which are maximal orders.

(Hipwood 2018) Any cg noetherian subalgebra of S that is an order in $Q_{\mathrm{gr}}(S)$ is an equivalent order to an iterated blowup of S at ≤ 2 points. Subalgebras which are maximal orders are classified.

What about overrings?

$\varphi : \mathrm{Bl}_p(\mathbb{P}^2) \to \mathbb{P}^2$	$\mathrm{Bl}_V(T) \subset T$
φ contracts L	How to contract L?

The geometric story:

Theorem (1)

(Castelnuovo) If X is a smooth projective surface containing a curve $L \cong \mathbb{P}^1$ with $\mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L) = 0$, then there is a smooth projective surface Y and a morphism $\varphi : X \to Y$ which contracts L to a point and is an isomorphism everywhere else. We have $X = \mathrm{Bl}_{\varphi(L)}(Y)$

Since $\varphi^{-1} : Y \dashrightarrow X$ is a blowup, we say $\varphi : X \to Y$ is a blowdown.

Theorem (2)

Any birational morphism $X \to Y$ of smooth projective surfaces is a composition of blowdowns.

Corollary (3)

If X contains no lines L with $\mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L) = 0$, then any birational $X \to Y$ is an isomorphism.

(We say X is a minimal model.)

In particular, $\mathbb{P}^2$ is a minimal model.

We seek a NC version of this geometry.

Fact: There is an automorphism σ of the elliptic curve E so that there is a ring homomorphism

$$\pi : S \to k(E)[t; \sigma]$$

with $\ker \pi = gS$. Further, σ is an infinite order translation.

$\pi(S)$ is a <u>twisted homogeneous coordinate ring</u>, as defined by Artin and Van den Bergh, and in their notation is written

$$\pi(S) = B(E, \mathcal{L}, \sigma)$$

where $\mathcal{L}$ is an invertible sheaf on E.

We have

$$\pi(T) = B(E, \mathcal{M}, \sigma^3) \qquad \text{for some } \mathcal{M}$$

Definition

A graded k-algebra R is an elliptic algebra if there is a central nonzerodivisor $g \in R_1$ so that

$$R/gR \cong B(E, \mathcal{N}, \tau)$$

for some elliptic curve E and infinite order translation τ (and some $\mathcal{N}$).

Elliptic algebras are cg noetherian domains.

If R is ellliptic, can blow up $p \in E$ to get $\widetilde{R} = \mathrm{Bl}_p(R) \subset R$, with $R/\widetilde{R} \cong \bigoplus_{n \geq 1} L(-n)$ as before.
For elliptic algebras, we have NC versions of the commutative results.

Theorem (1)

(Castelnuovo) If X is a smooth projective surface containing a curve $L \cong \mathbb{P}^1$ with $\mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L) = 0$, then there is a smooth projective surface Y and a morphism $\varphi : X \to Y$ which contracts L to a point and is an isomorphism everywhere else. We have $X = \mathrm{Bl}_{\varphi(L)}(Y)$

Since $\varphi^{-1} : Y \dashrightarrow X$ is a blowup, we say $\varphi : X \to Y$ is a blowdown.

Theorem (1NC)

(RSS 2016) Let R an elliptic algebra with associated elliptic curve E, and let L_R a line module with $\mathrm{Ext}^1_R(L, L) = 0$. Then there is an elliptic (thus noetherian) algebra R' with $R \subset R' \subset Q_{gr}(R)$ so that $R'/R \cong \bigoplus_{n\geq 1} L(-n)$.

We have $R = \mathrm{Bl}_p(R')$ for some $p \in E$.

R' is the blowdown of R at L.

Theorem (2)

Any birational morphism $X \to Y$ of smooth projective surfaces is a composition of blowdowns.

If R elliptic, define

$$R_{(g)} := R\langle h^{-1} : h \in R \smallsetminus gR \text{ homogeneous } \rangle$$

Theorem (2NC)

(RSS 2018) Let R be an elliptic algebra. Under a smoothness condition, any cg noetherian R' with $R \subseteq R' \subset R_{(g)}$ is obtained by blowing down finitely many lines L_i with $\mathrm{Ext}^1_R(L_i, L_i) = 0$.

The condition holds generically in examples (for blowups of T).

Corollary (3)

If X contains no lines L with $\mathrm{Ext}^1_X(\mathcal{O}_L, \mathcal{O}_L) = 0$, then any birational $X \to Y$ is an isomorphism. (X is a minimal model)

In particular, $\mathbb{P}^2$ is a minimal model.

Corollary (3NC)

(RSS 2018)

(a) *If R is cg noetherian with $T \subseteq R \subset T_{(g)}$ then $R = T$.*
(b) *Similar but more technical result without hypothesis that $R \subset T_{(g)}$*
(c) *If R is cg noetherian with $S \subseteq R \subset S_{(g)}$ then $R = S$.*

Remark

(i) *As a consequence of Corollary (3NC)(a) we obtain that if R is graded noetherian with $T \subsetneq R \subseteq Q_{\mathrm{gr}}(T)$ then* $\operatorname{GKdim} R \geq 4$*. Similar results hold for overrings of S.*

(ii) *It is easy to see that both the Corollary and (i) above fail for $k[x, y, z]$. Just consider $k[x, y, z, x^2/z]$.*

The analogy, so far:

geometry	algebra
$\mathbb{P}^2$	T
$p \in \mathbb{P}^2$	$V \in E$
$\varphi^{-1} : \mathbb{P}^2 \dashrightarrow \mathrm{Bl}_p(\mathbb{P}^2)$	$T \supset \mathrm{Bl}_V(T)$
exceptional line L	line module L
$\mathrm{Ext}^1(\mathcal{O}_L, \mathcal{O}_L) = 0$	$\mathrm{Ext}^1(L, L) = 0$
$\varphi : X \to Y$ contracting L	construct $R \subset R'$ with $R = \mathrm{Bl}_p(R')$ and $R'/R \cong \bigoplus_{n \geq 1} L(-n)$
$\mathbb{P}^2$ is a minimal model	T and S have few "nice" overrings

We continue the analogy by saying that T and S are minimal models.

We have classified cg noetherian R that are birational to T (or S) with either $R \subseteq T$ or $T \subseteq R$.

Question

Can we classify all cg noetherian R birational to S or to T?

Is geometric intuition helpful?

Thank you!