Localizable and Weakly Left Localizable Rings

V. V. Bavula (University of Sheffield) *

*1. V. V. Bavula, Left localizable rings and their characterizations, *J. Pure Appl. Algebra*, to appear, Arxiv:math.RA:1405.4552.

2. V. V. Bavula, Weakly left localizable rings, *Comm. Algebra*, **45** (2017) no. 9, 3798-3815.

talk-WLL-Rings(2015).tex

Aim:

- to introduce new classes of rings: **left localizable rings** and **weakly left localizable rings**, and

- to give several characterizations of them.

R is a ring with 1, R^* is its group of units,

$\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of R,

$Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the **left quotient ring** (the **classical left ring of fractions**) of R (if it exists),

$\mathrm{Ore}_l(R)$ is the set of **left Ore sets** S (i.e. for all $s \in S$ and $r \in R$: $Sr \cap Rs \neq \emptyset$),

$\mathrm{ass}(S) := \{r \in R \,|\, sr = 0$ for some $s \in S\}$, an ideal of R,

$\mathrm{Den}_l(R)$ is the set of **left denominator sets** S of R (i.e. $S \in \mathrm{Ore}_l(R)$, and $rs = 0$ implies $s'r = 0$ for some $s' \in S$),

$\mathrm{max.Den}_l(R)$ is the set of **maximal left denominator sets** of R (it is always a **non-empty** set).

$\mathfrak{l}_R := \bigcap_{S\in \text{max.Den}_l(R)} \text{ass}(S)$ is the **left localization radical** of R.

Theorem (B.'2014). If R is a left Noetherian ring then $|\text{max.Den}_l(R)| < \infty$.

A ring R is called a **left localizable ring** (resp. a **weakly left localizable ring**) if each nonzero (resp. non-nilpotent) element of R is a unit in some left localization $S^{-1}R$ of R (equiv., $r \in S$ for some $S \in \text{Den}_l(R)$).

Let $\mathcal{L}_l(R)$ be the **set of left localizable elements** and $\mathcal{NL}_l(R) := R\backslash\mathcal{L}_l(R)$ be the **set of left non-localizable elements** of R.

R is left localizable iff $\mathcal{L}_l(R) = R\backslash\{0\}$.

R is weakly left localizable iff $\mathcal{L}_l(R) = R\backslash\text{Nil}(R)$ where $\text{Nil}(R)$ is the set of nilpotent elements of R.

Characterizations of left localizable rings

- **Theorem** *Let R be a ring. The following statements are equivalent.*

  1. *The ring R is a left localizable ring with $n := |\text{max.Den}_l(R)| < \infty$.*

  2. *$Q_{l,cl}(R) = R_1 \times \cdots \times R_n$ where R_i are division rings.*

  3. *The ring R is a semiprime left Goldie ring with $\text{udim}(R) = |\text{Min}(R)| = n$ where $\text{Min}(R)$ is the set of minimal prime ideals of the ring R.*

  4. *$Q_l(R) = R_1 \times \cdots \times R_n$ where R_i are division rings.*

- **Theorem** *Let* R *be a ring with* $\max.\mathrm{Den}_l(R) = \{S_1, \dots, S_n\}$*. Let* $\mathfrak{a}_i := \mathrm{ass}(S_i)$*,*

$$\sigma_i : R \to R_i := S_i^{-1}R, \;\; r \mapsto \frac{r}{1} = r_i,$$

and $\sigma := \prod_{i=1}^n \sigma_i : R \to \prod_{i=1}^n R_i$*,* $r \mapsto (r_1, \dots, r_n)$*. The following statements are equivalent.*

1. *The ring* R *is a left localizable ring.*

2. $\mathfrak{l}_R = 0$ *and the rings* $R_1, \dots, R_n$ *are division rings.*

3. *The homomorphism* σ *is an injection and the rings* $R_1, \dots, R_n$ *are division rings.*

Characterizations of weakly left localizable rings

R is a **local ring** if $R \backslash R^*$ is an ideal of R ($\Leftrightarrow$ $R/\mathrm{rad}(R)$ is a division ring).

- **Theorem** *Let R be a ring. The following statements are equivalent.*

  1. *The ring R is a weakly left localizable ring such that*

     (a) $\mathfrak{l}_R = 0$,

     (b) $|\text{max.Den}_l(R)| < \infty$,

     (c) *for every $S \in \text{max.Den}_l(R)$, $S^{-1}R$ is a weakly left localizable ring, and*

     (d) for all $S, T \in \text{max.Den}_l(R)$ such that $S \neq T$, $\text{ass}(S)$ *is not a nil ideal modulo* $\text{ass}(T)$.

  2. $Q_{l,cl}(R) \simeq \prod_{i=1}^{n} R_i$ *where R_i are local rings with* $\text{rad}(R_i) = \mathcal{N}_{R_i}$.

  3. *$Q_l(R) \simeq \prod_{i=1}^{n} R_i$ where R_i are local rings with* $\text{rad}(R_i) = \mathcal{N}_{R_i}$.

Weakly left localizable rings rings have interesting properties.

- **Corollary** *Suppose that a ring R satisfies one of the equivalent conditions 1–3 of the above theorem. Then*

  1. $\text{max.Den}_l(R) = \{S_1, \dots, S_n\}$ *where* $S_i = \{r \in R \,|\, \frac{r}{1} \in R_i^*\}$.

  2. $\mathcal{C}_R = \bigcap_{S \in \text{max.Den}_l(R)} S$.

  3. $\text{Nil}(R) = \mathcal{N}_R$.

  4. $Q := Q_{l,cl}(R) = Q_l(R)$ *is a weakly left localizable ring with* $\text{Nil}(Q) = \mathcal{N}_Q = \text{rad}(Q)$.

  5. $\mathcal{C}_R^{-1}\mathcal{N}_R = \mathcal{N}_Q = \text{rad}(Q)$.

  6. $\mathcal{C}_R^{-1}\mathcal{L}_l(R) = \mathcal{L}_l(Q)$.

- **Theorem** *Let R be a ring, $\mathfrak{l} = \mathfrak{l}_R$, $\pi' : R \to R' := R/\mathfrak{l}$, $r \mapsto \bar{r} := r + \mathfrak{l}$. TFAE.*

  1. *R is a weakly left localizable ring s. t.*

     (a) *the map $\phi : \mathrm{max.Den}_l(R) \to \mathrm{max.Den}_l(R')$, $S \mapsto \pi'(S)$, is a surjection.*

     (b) $|\mathrm{max.Den}_l(R)| < \infty$,

     (c) for every $S \in \mathrm{max.Den}_l(R)$, $S^{-1}R$ is a weakly left localizable ring, and

     (d) for all $S, T \in \mathrm{max.Den}_l(R)$ such that $S \neq T$, $\mathrm{ass}(S)$ *is not a nil ideal modulo* $\mathrm{ass}(T)$.

  2. *$Q_{l,cl}(R') \simeq \prod_{i=1}^n R_i$ where R_i are local rings with* $\mathrm{rad}(R_i) = \mathcal{N}_{R_i}$, *$\mathfrak{l}$ is a nil ideal and $\pi'(\mathcal{L}_l(R)) = \mathcal{L}_l(R')$.*

  3. *$Q_l(R') \simeq \prod_{i=1}^n R_i$ where R_i are local rings with* $\mathrm{rad}(R_i) = \mathcal{N}_{R_i}$, *$\mathfrak{l}$ is a nil ideal and $\pi'(\mathcal{L}_l(R)) = \mathcal{L}_l(R')$.*

Criterion for a semilocal ring to be a weakly left localizable ring

A ring R is called a **semilocal ring** if $R/\text{rad}(R)$ is a semisimple (Artinian) ring.

The next theorem is a criterion for a semilocal ring R to be a weakly left localizable ring with $\text{rad}(R) = \mathcal{N}_R$.

- **Theorem** *Let R be a semilocal ring. Then the ring R is a weakly left localizable ring with $\text{rad}(R) = \mathcal{N}_R$ iff $R \simeq \prod_{i=1}^{s} R_i$ where R_i are local rings with $\text{rad}(R_i) = \mathcal{N}_{R_i}$.*