Reflections

The CASCADE CONJECTURE

Topology, the four color theorem and other PROBLEMS

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60th Birthdays

Let

$X = (x_1, x_2, x_3, \dots, x_n)$ points in

some vector space U over $\mathbb{R}$.

Let $r > 0$ be an integer

$T(X, r) = \{ y \in U :$ there

are r pairwise disjoint subsets

$J_1, J_2, \dots, J_r$ of $[n] = \{1, 2, \dots n\}$

such that

$y \in \operatorname{Conv}\{x_i : i \in J_1\} \cap \operatorname{conv}\{x_i : i \in J_2\} \dots$

$\cap \operatorname{conv}\{x_i : i \in J_r\}.$

$T(X, r)$ called : TVERBERG Points of order r

$T(X, 1)$

$\operatorname{conv}(x_1 \dots x_4)$

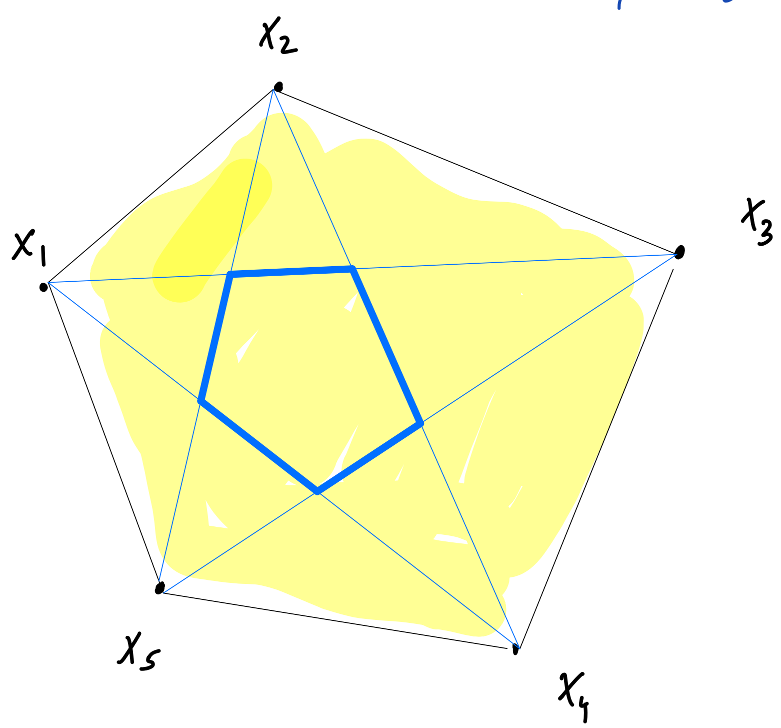
$t(X, r) = 1 + \dim T(X, r)$

$t(X, r) \geq 0 \quad t(X, r) = 0 \iff T(X, r) = \emptyset$

2)

$T(X, r) = \{ y \in U :$ there are r pairwise disjoint subsets $J_1, J_2, \dots, J_r$ of $[n] = \{1, 2, \dots, n\}$ such that

$y \in \operatorname{conv}\{x_i : i \in J_1\} \cap \operatorname{conv}\{x_i : i \in J_2\} \dots \cap \operatorname{conv}\{x_i : i \in J_r\}$.



$T(X, 1)$

$T(X, 2)$

$T(X, 3) = \emptyset$

$t(X, 1) = 2 + 1 = 3$

$t(X, 2) = 1 + 1 = 2$

$t(X, 3) = 0$

3/

THE CASCADE CONJECTURE

(Kalai, 74)

If

$$t(x,1)+t(x,2)+\dots+t(x,r) < |x|$$

then

$$t(x,r+1) > 0.$$

Another way to say it

$$\sum_{k\geq 1} t(x,k) \geqslant |X|.$$

4)

The first case of the CASCADE conjecture

If $t(X, 1) < |X|$

then $t(x, 2) > 0$

If $\dim \operatorname{conv}(X) + 1 < |X|$

then $t(X, 2) > 0$

This is RADON's theorem.

$\dim \operatorname{conv}(x) = \dim \operatorname{Aff}(X)$

if $|X| > \dim \operatorname{Aff}(X) + 1$

the $T(x, 2) \neq \emptyset$ i.e.

$\exists\ I, J \subset [n]$

$\operatorname{conv}\{x_i;\ i \in I\} \cap \operatorname{conv}\{x_j;\ j \in J\} \neq \emptyset.$

This is Radon's theorem.

5/

if $|X| > \dim \mathrm{Aff}(X) + 1$
then $x_1 \ldots x_n$ are affinly dependent

$\exists\ \lambda_1, \lambda_2 \ldots \lambda_n$ not all zero such that

$$\sum_{i=1}^{n} \lambda_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i = 0$$

Write: $I = \{i : \lambda_i \geq 0\}$ $J = \{j : \lambda_j < 0\}$

and $t = \sum_{i \in I} \lambda_i = \sum_{j \in J} (-\lambda_j)$, $t > 0$.

now $$\sum_{i \in I} \left(\frac{\lambda_i}{t}\right) x_i = \sum_{j \in J} \left(\frac{-\lambda_j}{t}\right) x_j$$

and $$\sum_{i \in I} \frac{\lambda_i}{t} = \sum_{j \in J} \frac{(-\lambda_j)}{t} = 1.$$

so $T(X, 2) \neq \emptyset$ $t(X, 2) > 0$.

Radon's thm 1920

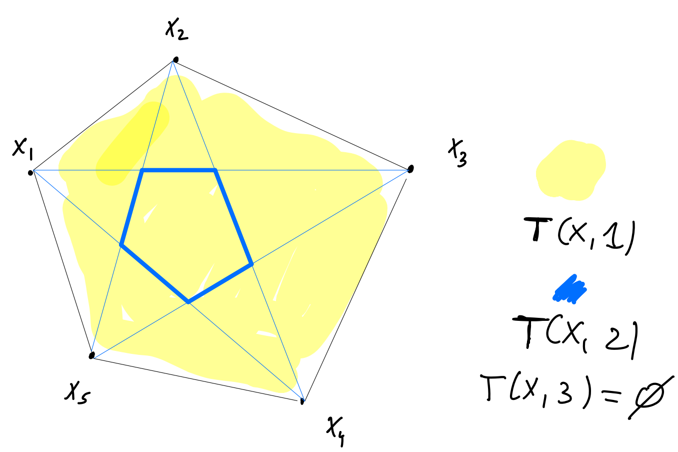
⑥ The first case of the CASCADE Conjecture: $t(X,1) < |X| \Rightarrow t(X,2) > 0$

this is $\boxed{\text{Radon's theorem.}}$

THE first open case of the CASCADE CONJECTURE

If $t(X,1) + t(X,2) < |X|$

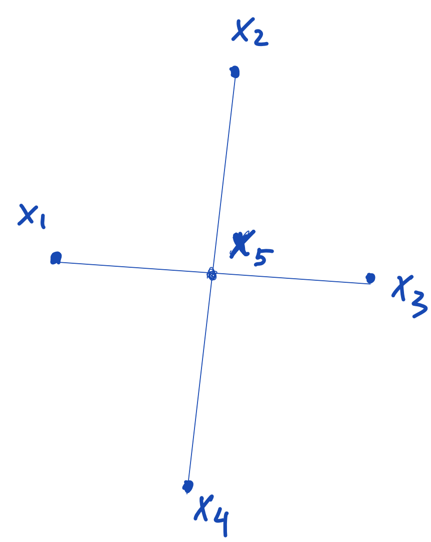
$\Longrightarrow t(X,3) > 0$ $\quad T(X,3) \neq \emptyset$



$t(X,1) = 2+1 = 3$ $\quad t(X,2) = 1+1 = 2$

$t(X,3) = 0$

$t(X,1) + t(X,2) = 5 = |X|$



$t(X,1) = 2+1 = 3$

$t(X,2) = 0+1 = 1$

$t(X,3) > 0$

$T(X,3) \neq \emptyset$

7

If $t(X,1)+t(X,2)<|X|$

$\Longrightarrow t(X,3)>0$

Corrolary: if $2t(X,1)<|X|$

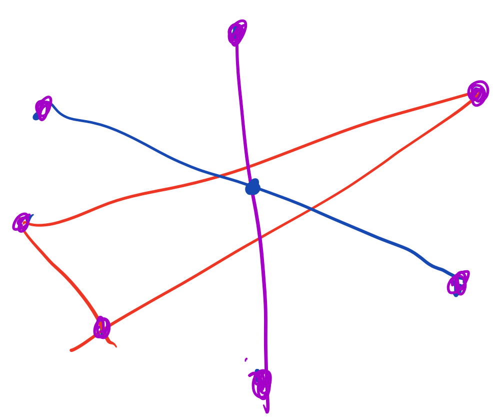
then $t(X,3)>0$

& If $|X|> 2\dim(X)+3$

$T(X,3)\neq\emptyset$

$X=7$ points in $\mathbb{R}^3$

$T(X,3)\neq\emptyset$.



The CASCADE CONJECTURE

(8) Full Generality

If $t(X,1) + t(X,2) + \dots + t(X,r) < |X|$

then $t(X, r+1) > 0$

<u>corollary</u>: (Tverberg's theorem):

If $|X| > r \cdot t(X,1) = r \cdot (\dim \mathrm{Aff}(X) + 1)$

then $T(X, r+1) \neq \emptyset$.

<u>Another formulation:</u> $\sum_{i \geq 1} t(X,i) \geq |X|$

True for $\dim \mathrm{Aff}(X) = 1$: easy

Theorem (Akiva KADARI, around 1980)

The cascade conjecture is true if

$\dim(X) = 2$.

9

Linear rather then affine formulation

Let $x_1 \cdots x_n$ ~~pod~~ point in real vector space U such that 0 is a vertex of $pos(x_1 \cdots x_n)$.

$$T^L(X, r) = \left\{ y \in U \quad y \in \begin{cases} pos\{x_i : i \in J_1\} \\ pos\{x_i : i \in J_2\} \\ \vdots \\ pos\{x_i : i \in J_r\} \end{cases} \right\}$$

$J_1 J_2 \cdots J_r$ pairwise disjoint subsets of $[n]$.

then $\sum \dim T^L(x, r) \geqslant |X|$.

Notions of cores

(10)

$$\mathrm{Core}_r(X) = \bigcap_{\substack{J \subset [n] \\ |J| < r}} \mathrm{conv}\{x_i : i \in [n] \setminus J\}.$$

$\mathrm{Core}_1(X) = \mathrm{conv}(X)$

Easy fact:

$$\mathrm{Core}_r(X) \supseteq T(X, r)$$

An ingredient in Kadari's proof in the plane $\mathrm{conv}\, T(X, r) = \mathrm{core}_r(X)$

(Not true in higher dimensions.)

$$T(X, r; i) = \bigcap_{|J| < i} T(X_{[n] \setminus J}, r)$$

$$T(X, r; i) \subseteq T(X, r + i - 1)$$

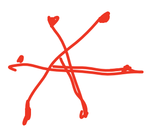
Order types

$(x_1, \dots, x_n)$ $\quad$ $(y_1, \dots, y_n)$

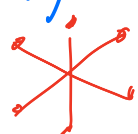
same order type if they have the same Radon partitions.

Namely: $\quad \mathrm{conv}\{x_i : i \in I\} \cap \mathrm{conv}\{x_j : i \in J\} \neq \emptyset$

for $I \cap J = \emptyset$ $\quad \Longleftrightarrow \mathrm{conv}\{y_i : i \in I\} \cap \mathrm{conv}\{y_i : i \in J\} \neq \emptyset$



$T(X,3) = \emptyset$



$T(X,3) \neq \emptyset$

$$s(X,r) = \min t(Y,r)$$

Y and X have the same order type.

A strong form Cascade conjecture is

$$s(X,1) + s(X,2) + \dots + s(X,r) < |X|$$

$$\Longrightarrow s(X, r+1) > 0.$$

ORDER TYPES & Universality. (12)

Erdős: There is a special order type.
configuration of m points in CYCLIC position which is UNIVERSAL
every collection of n pts in general position when n is large enough contains m pts in cyclic position.

For Tverberg points: The notion of order type is not developed
(But work of Perles and Sigron leads to a definition.)

Universal collection set of points where identified by Pór
(White, Bukh and Girash)...

ORDER TYPES

points in general position

$\Updownarrow$

certain determinants non-zero

$\Downarrow$

order type = signs of these polynomials

points in r-strong general position

$\Updownarrow$

certain polynomials are non zero

Higher-order types.
signs of these polynomials

Barany Kalai Por (21) Universality for collections of lines.

Problem (Perles): Given
$n = (d+1)(r-1)+1$ points in $\mathbb{R}^d$
and r numbers $a_1, a_2, \dots, a_r$ such that
$a_1 + \dots + a_r = n \qquad 1 \le a_i \le d+1 \quad i = 1, 2 \dots r$
is there a configuration of points
such that all Tverberg partitions
are of type $(a_1 \dots a_r)$?.
Answ (White): YES
Sierksma's Conjecture:
$\#$ Tverberg partitions $\geq ((r-1)!)^d$
White's examples · Equality
Bukh-Nivash more general examples.
Pór Universality

Enter topology

Rich connections between Helly type problems and topology.

- topological Helly
- topological Radon
- topological Tverberg.

THM

$f : \Delta^{d+1} \longrightarrow \mathbb{R}^d$ continuous

$\exists$ S, T disjoint faces of Δ^{d+2}

s.t. $f(S) \cap f(T) \neq \emptyset$.

Δ^{d+1} can be replaced by any $(d+1)$-polytope

any boundary of every $(d+1)$-dim convex body.

Bajmóczy and Bárány (79)

There is topological Tverberg theorem. (15)

when the number of parts prime, prime power.

Barany, Shlosman, Szücs (81)

Özaydin (87) Frick, Mabillard & Wagner

incorrect when # parts not prime power.

CAN WE USE TOPOLOGICAL MACHINARY TO FIND CONDITIONS FOR TVERBERG PARTITIONS INTO 3 PARTS WHEN THE NUMBER OF POINTS IS LESS THAN $2d+3$.

A vague topological extension of the CASCADE CONJECTURE

Vague Conjecture:

(several variant available)

X configuration of points $\mathbb{R}^d$

$P_2(X) \rightarrow$ collection of Radon partitions.

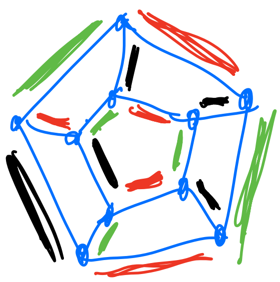
$\rightsquigarrow$ geometric complex.

$T(X,2) \rightarrow$ set of Radon points.

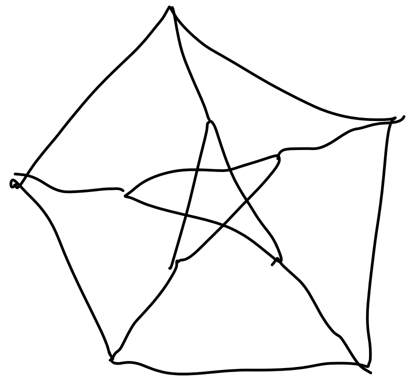
If map from $P_2(X) \rightarrow R_2(X)$ is topologically trivial (in some sense)

then $T(X,3) \neq \emptyset$.

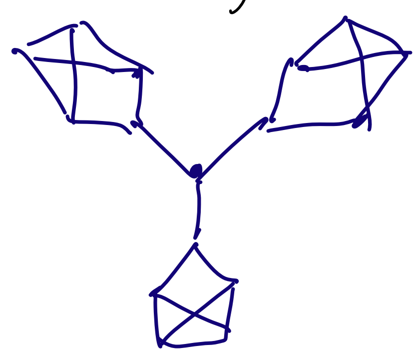
Coloring cubic graphs.



3-edge coloring



not 3-edge colorable.



4CT
(Four-Color theorem)

Every bridgeless planar cubic graph is 3-Edge Colorable.

Let G be a cubic graph $G=(V,E)$
$V=\{v_1 \dots v_n\}$

~~stw~~ associate to an edge $e=(v_i, v_j)$
the vector $(0,0 \dots \overset{i}{1} \dots 0 \dots \overset{j}{1}\ 0 \dots 0 \dots)$

This is a configuration ~~of~~ X of $\frac{3n}{2}$ points in $\mathbb{R}^d$ (actually $\mathbb{R}^{d-1}$).

__Observation__ G is 3-edge colorable iff $T(X,3) \neq \emptyset$. Shmuel Onn.

4CT: cubic bridgeless planar graphs are 3-edge colorable.

$\boxed{\text{Computational Complexity}}$

Apply a topological strengthening of the Cascade conjecture to show that

Every bridgeless planar graph is 3-edge colorable (4CT).

Find a proof that bipartite cubic graph is 3-edge colorable.

Happy Birthday

Ron an Yuval

Alon:

- List String ?

Lunch

hypertrees.

color/ut.

non-colorful.

Yuval's M.Sc Work.

Ron's Ph.D Work

group actions.

Ron
was my
first Ph.D
student

~~In~~ My
graduate years.

Pho.

mature

independent

Ron PhD
Yuval MSc

Ron stockholm89
Yuval K-L

Ron+Yuval 2000's

Chairperson Ron → Yuval

Ron was my first ph. D. student.
I was not sure that I'm sufficintly mature academically to have ph.D students and Ron was supervised together by Micha Perles who was my supervisor.
The question of my maturity was not settled since Ron himself was academically (and personally) very mature.

His ph.D path is similar to mine.

1) Result about hyperhees that was not included thesis.
2) Polytopes and complexes with group action. (stanley)

(Cubical complexes)

[Helly type theorem
[enumeration

☐ HTT

☐ hyp.t → colorful HTT

89 Stockholm Gelfand.

Responsibility maturity stability

~~Ron Aharoni~~

M. Sc.

jumpy

Yuval

Came with several research problems of his own, and solved them.

unexpected.

K-L polynomials

early 2000

Yuval Ron

Francesco Brenti

Reiner r Colored trees

Matroidal trees

Kazhdan

Lusztig.

Ron

Yuval

Yuval + Ron

Ron PhD

Yuval M.Sc

Ron Stockholm

some stories