Isotropic subspaces in symmetric composition algebras and Kummer subspaces in central simple algebras of degree 3

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Abstract. The maximal isotropic subspaces in split Cayley algebras were classified by van der Blij and Springer in Nieuw Archief voor Wiskunde VIII(3):158–169, 1960. Here we translate this classification to arbitrary composition algebras. We study intersection properties of such spaces in a symmetric composition algebra, and prove two triality results: one for two-D isotropic spaces, and another for isotropic vectors and maximal isotropic spaces. We bound the distance between isotropic spaces of various dimensions, and study the strong orthogonality relation on isotropic vectors, with its own bound on the distance. The results are used to classify maximal p-central subspaces in central simple algebras of degree $p = 3$. We prove various linkage properties of maximal p-central spaces and p-central elements. Analogous results are obtained for symmetric p-central elements with respect to an involution of the second kind inverting a third root of unity.

1. Introduction

A composition algebra is a (non-associative) algebra $(C, \star)$ over a field F, endowed with a non-degenerate quadratic "norm form" $\mathrm{N}: C \to F$, such that $\mathrm{N}(a \star b) = \mathrm{N}(a)\,\mathrm{N}(b)$.

Two special classes of composition algebras, which are of particular interest, have been fully classified. A composition algebra with unit is one of the following: the field itself, an étale quadratic extension, a quaternion algebra, or a Cayley algebra over F. In particular, the norm form is a Pfister form, so if there are isotropic elements, the form is hyperbolic.

A composition algebra is symmetric if the bilinar form associated to the norm is associative. One class of examples are the Okubo algebras, constructed from central simple algebras of degree 3 over the base field.

The maximal isotropic spaces in the split Cayley algebra were described by van der Blij and Springer [12], where the authors also explain the connection to geometric triality of points and two kinds of maximal spaces.

Kaplansky has shown that any composition algebra can be twisted into a unital one (see [6, Proposition 33.27]). We apply this technique to the classification of van der Blij and Springer, and obtain a complete classification of maximal isotropic

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spaces in a symmetric composition algebra. Similar results were obtained by the first named author in [7], using self-contained arguments which are similar to [12].

In the first part of this article (Sects. 2–8) we study isotropic subspaces in composition algebras. In Sect. 2 we give the essential background on composition algebras, mostly relying on [6, Sect. 33]. Section 3 describes the maximal isotropic spaces in an arbitrary composition algebra, and their intersections in case the algebra is symmetric. This section follows [12].

From Sect. 4 and on we assume C is a symmetric composition algebra. Any isotropic subspace of C is the intersection of two maximal subspaces, and we apply this fact to give a concrete classification of isotropic spaces of any possible dimension. In Sect. 5 we continue this line of research, by introducing the operators $\mathcal{L}$ and $\mathcal{R}$, defined as intersections of (left or right) maximal isotropic subspaces along a parameter space. It turns out that $\mathcal{L}$ acts on the space of isotropic 2-D spaces.

In Sect. 6 we show that $\mathcal{L} \circ \mathcal{L} \circ \mathcal{L}$ is the identity on 2-D isotropic spaces. This provides an analog to the geometric triality studied in [12]. The relation with triality of automorphisms of spin groups for symmetric composition algebras, as described in [2], is yet to be investigated.

In Sect. 7 we study distances between various types of isotropic spaces. For example, we show that for every isotropic vector x and every maximal isotropic subspace V, there are a maximal isotropic subspace $V' \ni x$ and an isotropic vector $x' \in V \cap V'$.

Section 8 is devoted to the 'strong orthogonality' relation—we prove that every two isotropic elements are connected by a chain of length 4, in which every two elements are strongly orthogonal. This statement has many variations, and in particular it allows us to deduce a common slot lemma for central simple algebras of degree 3, which was discovered by Rost [10].

The second part (Sects. 9–15) is devoted to applications to central simple algebras. Let A be a central simple algebra of prime degree p over a field F. An non-central element $x \in A$ is 'p-central' if $x^p \in F$.

Following the recent work of Raczek ([8]; see also [9]), who studied 3-D spaces of 3-central elements in cyclic algebras of degree 3, we consider in this part the maximal linear subspaces of 3-central elements in a symbol algebra of degree 3 (when there are roots of unity), which are 4-D, and the interaction between them.

The classification is based on the following idea of J.-P. Tignol: under the Okubo product, the space of zero-trace elements in an algebra A of degree 3 becomes a (non-unital, symmetric) composition algebra. We can thus translate the analysis of maximal isotropic subspaces in symmetric composition algebras from the first part, to 3-central spaces in symbol algebras of degree 3.

Section 9 introduces the 'standard' p-central spaces, those of the form $Fx + F[x]y$ where $yxy^{-1} = \rho x$. In Sect. 10 we describe the Okubo product, and show (in Theorem 10.2) that for $p = 3$, every maximal p-central subspace is standard.

In Sect. 11 we study the Okubo product further, showing for example that the operation defined via Kaplansky's unital shift has a canonical multiplication table, which is monomial and independent of the algebra A as well as of the choice of a standard pair of generators.

The main purpose of Sect. 12 is to study standard pairs of generators in a given 3-central subspace: if the space is maximal, the variety of standard pairs of generators inside it is (geometrically) a blowup of $\mathbb{P}^3F$ at a point. A 3-D subspace can be presented uniquely in the form $Fy + Fxy^2 + Fz$ where $yxy^{-1} = \rho x$ and $Fz \subseteq Fx + Fx^2y$. These spaces come in three flavors, depending on the amount of standard pairs of generators they contain. Finally in Sect. 13 we show that maximal 3-central spaces are linked by chains, where the intersection of every two neighbors has dimension 2 or 3. We also show how the slot lemma of M. Rost for cyclic algebras of degree 3 follows from a more general result on symmetric composition algebras. This was done, directly for Okubo algebras and with an explicit description of the chains, in [5].

In the final sections we assume F does not have third roots of unity. Then one can construct a composition algebra from an algebra of degree 3 over $K = F[\rho]$, endowed with an involution of the second kind whose fixed subfield is F. We show (Theorem 14.1) that such algebras have symmetric standard pair of generators; in particular, they all have the form $(\alpha, \beta)_K$ for $\alpha, \beta \in F$. We are then able to classify in Sect. 15 the maximal 3-central symmetric subspaces, and prove a slot lemma for standard pairs of symmetric generators.

2. Background on composition algebras

In the first part, Sects. 2–8, F is an arbitrary field.

2.1. Composition algebras

A **composition algebra** is a (non-unital, non-associative) algebra $(C, \star)$ over a field F, with a non-singular quadratic form $\mathrm{N} : C \to F$ such that $\mathrm{N}(a \star b) = \mathrm{N}(a)\,\mathrm{N}(b)$. The composition is **symmetric** if the associated bilinear form $\mathrm{B}(a, b) = \mathrm{N}(a + b) - \mathrm{N}(a) - \mathrm{N}(b)$ is associative, namely

$$\mathrm{B}(a \star b, c) = \mathrm{B}(a, b \star c). \tag{2.1}$$

This is the case iff

$$(a \star b) \star a = a \star (b \star a) = \mathrm{N}(a)b \tag{2.2}$$

for every $a, b \in C$ [6, Lemma 34.1] (the equality at the left-hand side is the "flexible" identity). The opposite of a composition algebra is again a composition algebra (with the same N and B), which justifies taking the mirror image of various statements, when needed.

Remark 2.1. Multilinearization of (2.2) yields the identities

$$(a \star b) \star c + (c \star b) \star a = \mathrm{B}(a, c)b \tag{2.3}$$

$$a \star (b \star c) + c \star (b \star a) = \mathrm{B}(a, c)b. \tag{2.4}$$

Remark 2.2. The subalgebra generated by an element a in a symmetric composition algebra is $Fa + F(a \star a)$. Indeed $(a \star a) \star a = a \star (a \star a) = \mathrm{N}(a)a$ by (2.2) and

$$(a \star a) \star (a \star a) = \mathrm{B}(a, a \star a)a - \mathrm{N}(a)(a \star a) \tag{2.5}$$

by (2.3). This shows that $Fa + F(a \star a)$ is closed under multiplication.

2.2. *Unital composition algebras*

A composition algebra $(C, \diamond, \mathrm{N})$ with unit e is endowed with a standard involution

$$a \mapsto \overline{a}, \tag{2.6}$$

defined by

$$\overline{a} + a = \mathrm{B}(a, e)e \tag{2.7}$$

(Thus $\overline{e} = e$).

Such algebras have dimension 1, 2, 4 or 8. In the case of dimension 8 the algebra is necessarily a Cayley algebra [6, Theorem 33.17] (i.e. constructed from a quaternion algebra via the Cayley-Dickson construction). Unital composition algebras (of dimension > 2) are never symmetric.

2.3. *Unital translation*

Let $(C, \star, N)$ be any composition algebra. Let u be any element with $\mathrm{N}(u) = 1$ (for example take $u = \mathrm{N}(v)^{-1}(v \star v)$ where v is any element with $\mathrm{N}(v) \neq 0$). Since the multiplication maps $R_u : a \mapsto a \star u$ and $L_u : a \mapsto u \star a$ are isometries of (C, N), they are invertible, and one can define a new operation

$$a \diamond b = (R_u^{-1}a) \star (L_u^{-1}b), \tag{2.8}$$

for which the following diagram commutes:

$$\begin{array}{ccc} C \times C & \xrightarrow{\ \star\ } & C \\ {\scriptstyle R_u \times L_u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C \times C & \xrightarrow{\ \diamond\ } & C \end{array}$$

Then $(C, \diamond)$ is a composition algebra with $e = u \star u$ as a unit (the construction is due to Kaplansky; see [6, Proposition 33.27]), retaining the same norm form. In particular the associated bilinear form remains the same.

When $(C, \star)$ is symmetric we have (2.2) at our disposal, and then R_u and L_u are inverse to each other. In this case (2.8) can be rewritten in the form

$$a \diamond b = (u \star a) \star (b \star u). \tag{2.9}$$

3. Isotropic subspaces

The norm form of a Cayley algebra is a Pfister form, so it is either anisotropic or hyperbolic. When the form is hyperbolic, the maximal hyperbolic subspaces are of dimension four. Since the unital translation of Sect. 2.3 preserves the form, the same is true for any composition algebra.

The purpose of this section is to describe the main results of [12] concerning isotropic subspaces of the split Cayley algebra, and translate them to arbitrary composition algebras, in particular symmetric ones. We minimize the amount of independent computation, and rely on [12] whenever possible. As mentioned in the introduction, the results of this section can also be obtained by direct computation, as in [7].

3.1. Isotropic subspaces of a Cayley algebra

In any composition algebra $(C, \star)$, if x is isotropic then $C \star x$ and $x \star C$ are isotropic spaces. The maximal isotropic subspaces of the split Cayley algebra $(C, \diamond, \mathrm{N}, e)$ were classified by van der Blij and Springer in [12, Theorem 3], with a pleasantly simple formulation: they are precisely the subspaces of the form $C \diamond x$ or $x \diamond C$, ranging over isotropic vectors x.

Theorem 3.1. *Let $(C, \star, \mathrm{N})$ be an arbitrary composition algebra. Then every maximal isotropic subspace of C is of the form $x \star C$ or $C \star x$ for x isotropic; these spaces have dimension* 4.

Proof. Let $u \in C$ be an element with $\mathrm{N}(u) = 1$, and define the operation $\diamond$ by (2.8). Then $(C, \diamond, \mathrm{N})$ is a Cayley algebra. Therefore, every maximal isotropic subspace V has the form $x \diamond C$ or $C \diamond x$ for some x with $\mathrm{N}(x) = 0$. The dimension is $4 = \frac{1}{2}\dim(C)$ since the norm form is hyperbolic.

However, since $L_u^{-1}C = R_u^{-1}C = C$, and since $L_u^{-1}x$ and $R_u^{-1}x$ range over all isotropic elements when x does, V must have the form $x' \star C$ or $C \star x'$ for a suitable isotropic $x' \in C$. □

A maximal isotropic subspace has a unique (projective) generator:

Proposition 3.2. *Let x, x' be isotropic vectors. Then $x \star C = x' \star C$ iff $Fx = Fx'$; likewise for left spaces.*

Proof. By (2.8), $R_u(a) \diamond L_u(b) = a \star b$, so $x \star C = R_u(x) \diamond L_u(C) = R_u(x) \diamond C$. If $x \star C = x' \star C$ we get $R_u(x) \diamond C = R_u(x') \diamond C$, which by [12, Theorem 4] implies $F \cdot R_u(x) = F \cdot R_u(x')$, hence $Fx = Fx'$. □

It is possible to translate the results of [12] on intersection of maximal isotropic spaces to arbitrary composition algebras. However the outcome is not very enlightening, so we restrict the rest of the discussion to symmetric composition algebras.

3.2. Multiplication operators in the symmetric case

Throughout the rest of this section we assume $(C, \star)$ is a symmetric composition algebra, $u \in C$ is a fixed element with $\mathrm{N}(u) = 1$, and $\diamond$ is defined by (2.9). Let L_x and R_x denote the multiplication from left or right by x in C. In particular $C \star x = \mathrm{Im}(R_x)$ and $x \star C = \mathrm{Im}(L_x)$. Although $x \star C$ and $C \star x$ are not one-sided ideals (the algebra is not associative), we say that $x \star C$ is a space of the right kind, and $C \star x$ a space of the left kind.

Remark 3.3. If $a, b \in C$ satisfy $a \star b = 0$ where $a, b \neq 0$, then a and b are both isotropic. Indeed $\mathrm{N}(a)b = (a \star b) \star a = 0$ implies $\mathrm{N}(a) = 0$, and similarly for b.

Proposition 3.4. *Let $x \neq 0$ be an isotropic element in a symmetric composition algebra $(C, \star)$. Then* $\mathrm{Ker}(L_x) = \mathrm{Im}(R_x)$ *and* $\mathrm{Ker}(R_x) = \mathrm{Im}(L_x)$.

Proof. By (2.2) the composition $L_x R_x = R_x L_x$ is multiplication by $\mathrm{N}(x) = 0$, so $\mathrm{Im}(R_x) \subseteq \mathrm{Ker}(L_x)$; but if $x \star a = 0$ then a is isotropic by Remark 3.3, showing that $\mathrm{Ker}(L_x)$ is an isotropic space, hence $\dim(\mathrm{Ker}(L_x)) \leq 4$ and $\mathrm{Ker}(L_x) = \mathrm{Im}(R_x)$. The argument for $\mathrm{Ker}(R_x) = \mathrm{Im}(L_x)$ is the same. □

Corollary 3.5. *For any isotropic vectors x and y, $x \in C \star y$ iff $y \in x \star C$.*

Proof. Indeed, $x \in \mathrm{Im}(R_y)$ iff $x \in \mathrm{Ker}(L_y)$ iff $y \star x = 0$ iff $y \in \mathrm{Ker}(R_x)$ iff $y \in \mathrm{Im}(L_x)$. □

3.3. Intersection of maximal isotropic spaces

We now consider various intersections of the maximal isotropic subspaces appearing in Theorem 3.1, again following [12].

Let us record several easy observations which will be used frequently. Notice that by (2.2), we have

$$u \star (a \star u) = (u \star a) \star u = a. \tag{3.1}$$

Remark 3.6. For any isotropic $x, x' \in C$:

(1) By (2.9) we have that

$$C \diamond x = C \star (x \star u), \quad x \diamond C = (u \star x) \star C$$

and

$$C \star x = C \diamond (u \star x), \quad x \star C = (x \star u) \diamond C.$$

(2) $F \cdot (x \star u) = F \cdot (x' \star u)$ iff $F \cdot (u \star x) = F \cdot (u \star x')$ iff $Fx = Fx'$. (Left and right multiplication by u are invertible linear maps).
(3) $\mathrm{B}(x \star u, x' \star u) = \mathrm{B}(u \star x, u \star x') = \mathrm{B}(x, x')$ (by (2.1) and (3.1)).

The intersection of maximal isotropic spaces of the same kind has even dimension:

Proposition 3.7. *Let x, x' be linearly independent isotropic vectors.*

(1) *If* $\mathrm{B}(x, x') = 0$ *then* $V = x \star C \cap x' \star C$ *is equal to* $x \star (C \star x') = x' \star (C \star x)$*, which has dimension* 2*; otherwise,* $V = 0$.
(2) *If* $\mathrm{B}(x, x') = 0$ *then* $V = C \star x \cap C \star x'$ *is equal to* $(x' \star C) \star x = (x \star C) \star x'$*, which has dimension* 2*; otherwise,* $V = 0$.

Proof. It suffices to prove (1). Since $x \star C = (x \star u) \diamond C$ and likewise for x', we apply [12, Theorem 4] to the intersection $(x \star u) \diamond C \cap (x' \star u) \diamond C$; thus, the intersection has dimension 2 iff $\mathrm{B}(x, x') = \mathrm{B}(x \star u, x' \star u) = 0$, by Remark 3.6.(3), and otherwise the intersection is zero.

We now compute V when $\mathrm{B}(x, x') = 0$. First note that by (2.4) (with $a = x$ and $c = x'$), $x \star (C \star x') = x' \star (C \star x)$, and in particular $x \star (C \star x') \subseteq V$.

Now suppose $v \in V$, then $v = x \star a = x' \star b$ for suitable $a, b \in C$. Taking into account the non-degeneracy of B and the fact that x and x' are linearly independent, we may find $c \in C$ such that $B(x, c) = 1$ and $B(x', c) = 0$. From (2.4) we have $a = B(x, c)a = (x \star a) \star c + (c \star a) \star x$, but $x \star (C \star x) = 0$, so $v = x \star a = x \star ((x \star a) \star c + (c \star a) \star x) = x \star ((x \star a) \star c) = x \star (v \star c)$. On the other hand, from (2.3) we get $(x' \star b) \star c = -(c \star b) \star x'$, so that $v = x \star (v \star c) = x \star ((x' \star b) \star c) = -x \star ((c \star b) \star x') \in x \star (C \star x')$. Thus $V = x \star (C \star x') = x' \star (C \star x)$.

The formula in [12] is $x \diamond C \cap x' \diamond C = x \diamond (\overline{x'} \diamond C) = x' \diamond (\overline{x} \diamond C)$, where the involution is defined in (2.7). Direct substitution in this gives the somewhat disappointing formula $(x \star C) \cap (x' \star C) = x \star (((u \star \overline{x' \star u}) \star C) \star u)$, from which we conclude that

$$x \star (((u \star \overline{x' \star u}) \star C) \star u) = x \star (C \star x')$$

whenever $\mathrm{B}(x, x') = 0$; note that the involution in the left-hand side depends on the choice of u. □

We will denote by $P^{\perp}$ the space orthogonal to P with respect to B. In particular $x^{\perp}$ denotes the 7-D subspace $\{a \in C : \mathrm{B}(a, x) = 0\}$. As usual, $x \in x^{\perp}$ iff x is isotropic.

The intersection of maximal isotropic spaces of opposing kinds has dimension 1 or 3:

Proposition 3.8. *Let x, x' be isotropic vectors. The intersection of $x \star C$ and $C \star x'$ is:*

(1) *the* 1*-D space* $F \cdot (x \star x')$ *if* $x \star x' \neq 0$*; and*
(2) *the* 3*-D space* $x \star x'^{\perp} = x^{\perp} \star x'$ *if* $x \star x' = 0$.

Proof. Let us quote [12, Theorem 5]:

(1) If $x \diamond x' \neq 0$ then $(x \diamond C) \cap (C \diamond x') = F \cdot (x \diamond x')$;
(2) If $x \diamond x' = 0$, $x \diamond C \cap C \diamond x'$ is the 3-D space $x \diamond x'^{\perp}$.

The identification of $(x \star C) \cap (C \star x')$ as $F \cdot (x \star x')$ or $x \star x'^{\perp}$ follows by taking $x \star u$ for x and $u \star x'$ for x'. Applying this to the opposite algebra (and replacing x and x'), when $x \star x' = 0$, we obtain the equality $x \star x'^{\perp} = x^{\perp} \star x'$. □

Summarizing, the intersection of spaces of the same kind is even dimensional, and the intersection of spaces of different kinds is odd dimensional.

Corollary 3.9. *A maximal isotropic subspace has a well defined kind.*

4. Prescribed intersection of maximal subspaces

We now study the ways in which a given isotropic subspace in a symmetric composition algebra C can be presented as the intersection of maximal subspaces. First note that such a presentation always exist:

Proposition 4.1. *In any hyperbolic space, every isotropic subspace U is the intersection of two maximal isotropic spaces, one of which may be chosen arbitrarily.*

Proof. Let V_1 be any maximal isotropic space containing U, and write $A = V_1 \oplus W$ where W is a maximal isotropic space, using the fact that the form is non-degenerate.

Let $W_0 = W \cap U^{\perp}$ and take $V_2 = U \oplus W_0$, which is an orthogonal sum of isotropic spaces, and so isotropic. Since $A = W + V_1 \subseteq W + U^{\perp}$, $\dim(W_0) = \dim(U^{\perp}) + \dim(W) - \dim(A) = \dim(W) - \dim(U)$, so $\dim(V_2) = \dim(W)$ and V_2 is maximal. Clearly $U \subseteq V_1 \cap V_2$, and by modularity $V_1 \cap V_2 = V_1 \cap (W_0 + U) = (V_1 \cap W_0) + U \subseteq (V_1 \cap W) + U = U$. □

The first application is a unique presentation for 3-D spaces. The presentations of 2-D spaces are classified, to some extent, in Proposition 5.4.

Proposition 4.2. *A 3-D isotropic subspace $U \subseteq C$ is contained in a unique maximal isotropic space of each kind.*

In particular, every such U can be presented as $x \star x'^{\perp} = x^{\perp} \star x'$ for x, x' isotropic with $x \star x' = 0$, in a (projectively) unique way.

Proof. Write $U = V_1 \cap V_2$ where V_1 and V_2 are maximal. They must be of different kinds, and U cannot be contained in any other maximal space, since the intersection of two maximal spaces of the same kind is of dimension at most 2 by Proposition 3.7. Writing $V_1 = x \star C$ and $V_2 = C \star x'$, the space has the form $x \star x'^{\perp}$ by Proposition 3.8, so the uniqueness follows from Proposition 3.2. □

Remark 4.3. If $x \star x' \neq 0$ then $x \star x'^{\perp} = x \star C$, and $x^{\perp} \star x' = C \star x'$. Indeed, for some $a \in C$, $\mathrm{B}(a \star x, x') = \mathrm{B}(a, x \star x') \neq 0$, which shows that $\mathrm{Ker}(L_x) = \mathrm{Im}(R_x)$ is not contained in $x'^{\perp}$. Therefore $\dim(\mathrm{Ker}(L_x) \cap x'^{\perp}) = 3$, and $x \star x'^{\perp} = L_x(x'^{\perp})$ has dimension $7 - 3 = 4$; but clearly $x \star x'^{\perp} \subseteq x \star C$. Likewise for $x^{\perp} \star x'$.

5. Parameterized intersections

We turn to the intersection of an arbitrary number of maximal isotropic spaces. Our point of departure is the following observation.

Proposition 5.1. *Let P_0 be any set of isotropic elements. The intersections $\cap_{x \in P_0}(C \star x)$ and $\cap_{x \in P_0}(x \star C)$ depend only on the linear span of P_0.*

Proof. Indeed, every maximal isotropic space is self-orthogonal, and $\cap_{x \in P_0}(C \star x)$ is the space orthogonal to

$$(\cap_{x \in P_0}(C \star x))^{\perp} = \sum_{x \in P_0}(C \star x)^{\perp} = \sum_{x \in P_0}(C \star x) = C \star \operatorname{span} P_0.$$

Similarly for $\cap_{x \in P_0}(x \star C)$. □

By Proposition 3.7, all generators participating in a non-zero intersection of spaces of the same kind are mutually orthogonal. Therefore, the most general intersections to consider are

$$\mathcal{L}(P) = \cap_{x \in P}(C \star x) \tag{5.1}$$

and

$$\mathcal{R}(P) = \cap_{x \in P}(x \star C),$$

when P is an isotropic space. As seen in Proposition 5.1,

$$\begin{aligned} \mathcal{L}(P) &= (C \star P)^{\perp} \\ &= \{z \in C : \ \mathrm{B}(C \star P, z) = 0\} \\ &= \{z : \ \mathrm{B}(C, P \star z) = 0\} \\ &= \{z : \ P \star z = 0\} \\ &= \mathrm{Ann}_r(P), \end{aligned} \tag{5.2}$$

and likewise $\mathcal{R}(P) = \mathrm{Ann}_{\ell}(P)$.

Proposition 5.2. *For every isotropic subspace P,*

(1) *$w \in \mathcal{L}(P)$ iff $P \subseteq w \star C$,*
(2) *$w \in \mathcal{R}(P)$ iff $P \subseteq C \star w$;*

and also, for every isotropic P and P',

(3) *$P' \subseteq \mathcal{L}(P)$ iff $P \subseteq \mathcal{R}(P')$.*

Proof. By Corollary 3.5, $w \in \mathcal{L}(P)$ iff $w \in C \star x$ for every $x \in P$, iff $x \in w \star C$ for every x in P, iff $P \subseteq w \star C$. Likewise for $\mathcal{R}(P)$. As for (3), $P' \subseteq \mathcal{L}(P)$ iff $P \subseteq w \star C$ for every $w \in P'$, iff $P \subseteq \mathcal{R}(P')$. □

Let $\mathcal{P}_k$, $k = 1, 2, 3$, denote the classes of k-dimensional isotropic spaces in C, and likewise let $\mathcal{P}_4$ and $\mathcal{P}_{4'}$ denote the classes of left and right maximal isotropic spaces, respectively.

Proposition 5.3. *The action of the operators $\mathcal{L}$ and $\mathcal{R}$ on isotropic spaces is as follows.*

(1) *For any isotropic vector* x, $\mathcal{L}(Fx) = C \star x$ *and* $\mathcal{R}(Fx) = x \star C$;
(2) *For* $P \in \mathcal{P}_2$, $\mathcal{L}(P) = (x \star C) \star x'$ *and* $\mathcal{R}(P) = x \star (C \star x')$, *which are* 2*-D, for any basis* $\{x, x'\}$ *of* P.
(3) *If* $x \star x' = 0$ *then* $\mathcal{L}(x \star x'^{\perp}) = Fx$ *and* $\mathcal{R}(x \star x'^{\perp}) = Fx'$.
(4) *For an isotropic vector* x, $\mathcal{L}(C \star x) = 0$ *and* $\mathcal{R}(C \star x) = Fx$; *likewise* $\mathcal{L}(x \star C) = Fx$ *and* $\mathcal{R}(x \star C) = 0$.

In particular, for P *of dimension* 1, 2, *or* 3, $\mathcal{L}(P)$ *and* $\mathcal{R}(P)$ *have dimensions* 4, 2 *and* 1, *respectively.*

Proof. For $P = Fx$ only one space participates in the intersection. The case $\dim P = 2$ is in Proposition 3.7. For $\dim P = 3$ (where we apply Proposition 4.2), note that $Fw \subseteq \mathcal{L}(P)$ iff $P \subseteq w \star C$ by Proposition 5.2, and by Proposition 4.2 this Fw is unique; moreover when $P = x \star x'^{\perp}$, necessarily $Fw = Fx$. Likewise for $\mathcal{R}(P)$.

Finally, consider the space $P = C \star x$. As in the previous case, $Fw \subseteq \mathcal{L}(C \star x)$ iff $C \star x \subseteq w \star C$, which implies $w = 0$; and $Fw \subseteq \mathcal{R}(C \star x)$ iff $C \star x \subseteq C \star w$, which is the case iff $Fw = Fx$ by Proposition 3.2. The same argument takes care of $\mathcal{R}(P)$. □

Corollary 5.4. *For any* 2*-D isotropic space* P, $\mathcal{LR}(P) = \mathcal{RL}(P) = P$. *Moreover* $P = x \star (C \star x')$ *iff* $\{x, x'\}$ *span* $\mathcal{L}(P)$, *and* $P = (y \star C) \star y'$ *iff* $\{y, y'\}$ *span* $\mathcal{R}(P)$.

Proof. By Proposition 5.2.(3), for any 2-D isotropic spaces P' and P'', $P' = \mathcal{L}(P'')$ iff $P'' = \mathcal{R}(P')$, so in particular $\mathcal{LR}(P) = \mathcal{RL}(P) = P$. It follows that P can be expressed in the forms $P = \mathcal{L}(P')$ and $P = \mathcal{R}(P'')$ in a unique way. The rest follows from Proposition 5.3.(2). □

It follows from Proposition 5.3 and Corollary 5.4 that any isotropic space of dimension different than 3 is of the form $\mathcal{L}(P)$ and $\mathcal{R}(P')$ for suitable spaces P and P'. For dimension 3 we have:

Corollary 5.5. *Let* U *be a* 3*-D isotropic space. Then* $\mathcal{LR}(U)$ *and* $\mathcal{RL}(U)$ *are the unique left and right maximal isotropic spaces containing* U, *respectively. Thus*

$$U = \mathcal{RL}(U) \cap \mathcal{LR}(U).$$

Proof. Write $U = x \star x'^{\perp}$. Then $\mathcal{L}(U) = Fx$, so $\mathcal{RL}(U) = x \star C$; and $\mathcal{R}(U) = Fx'$, so $\mathcal{LR}(U) = C \star x'$. □

6. Triality

6.1. Triality of 2*-D spaces*

Concerning the interaction between isotropic spaces, one may ask: When is a given 2-D isotropic space contained in a given 3-D space?

What are the 3-D spaces containing a given 2-D isotropic space P? Recall from Proposition 4.2 that a 3-D space has a unique presentation in the form $x \star x'^{\perp}$.

Proposition 6.1. *Let P be a 2-D isotropic space. Then $P \subseteq x \star x'^{\perp}$ iff $x \in \mathcal{L}(P)$ and $x' \in \mathcal{R}(P)$.*

Proof. Writing $x \star x'^{\perp} = x \star C \cap C \star x'$, this is immediate from Proposition 5.2. □

Theorem 6.2. *For any 2-D space P, $\mathcal{L}(\mathcal{L}(\mathcal{L}(P))) = P$.*

Proof. Let $x \in \mathcal{L}(P)$ and $x' \in \mathcal{R}(P)$. By Proposition 6.1 the dimension of $x \star C \cap C \star x'$ is at least 2, so by Proposition 3.8, $x \star x' = 0$. Thus $\mathcal{L}(P) \star \mathcal{R}(P) = 0$, and $\mathcal{R}(P)$ is contained in the right annihilator of $\mathcal{L}(P)$. By (5.2), it follows that $\mathcal{R}(P) \subseteq \mathcal{L}(\mathcal{L}(P))$, but the dimensions are equal by Proposition 5.3.(2), so $\mathcal{R}(P) = \mathcal{L}(\mathcal{L}(P))$, and the result follows from Corollary 5.4. □

By Corollary 5.4, we also have $\mathcal{R}(\mathcal{R}(\mathcal{R}(P))) = P$.

6.2. *Geometric triality*

Following the section on geometric triality in [12], we now define the following graph structure on the vertices $\mathcal{P}_1 \cup \mathcal{P}_4 \cup \mathcal{P}_{4'}$:

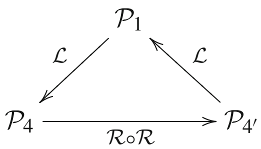
- $K, K' \in \mathcal{P}_1$ are connected by an edge if $K + K' \in \mathcal{P}_2$;
- $U, U' \in \mathcal{P}_4$ are connected by an edge if $U \cap U' \in \mathcal{P}_2$;
- $U, U' \in \mathcal{P}_{4'}$ are connected by an edge if $U \cap U' \in \mathcal{P}_2$;
- $U \in \mathcal{P}_4$ is connected by an edge with $U' \in \mathcal{P}_{4'}$ if $U \cap U' \in \mathcal{P}_3$;
- $K \in \mathcal{P}_1$ is connected by an edge with $U \in \mathcal{P}_4$ or $U \in \mathcal{P}_{4'}$ if $K \subseteq U$.

Evidently in the first three cases, every edge connecting vertices of the same class has a unique label from $\mathcal{P}_2$.

Equivalently (following Propositions 3.4, 3.7 and 3.8), the graph can also be described as follows.

$$(Fx, Fx'), (C \star x, C \star x'), (x \star C, x' \star C) \text{ are edges if } \mathrm{B}(x, x') = 0;$$
$$(C \star x, x' \star C), (x \star C, Fx'), (Fx, C \star x') \text{ are edges if } x' \star x = 0.$$

Let us consider the triality map ρ defined by the diagram below.



By Remark 5.3, ρ maps $Fx \mapsto C \star x \mapsto x \star C \mapsto Fx$, and thus has order three.

Proposition 6.3. *If two vertices t, t' of the same class are connected, then so are $\rho(t)$ and $\rho(t')$. Moreover when the vertices at the left-hand side of the equalities below are connected, we have:*

$$\rho(Fx) \cap \rho(Fx') = \mathcal{L}(Fx + Fx'),$$
$$\rho(C \star x) \cap \rho(C \star x') = \mathcal{L}(C \star x \cap C \star x'),$$
$$\rho(x \star C) + \rho(x' \star C) = \mathcal{L}(x \star C \cap x' \star C).$$

Proof. Fx and Fx' are connected when $P = Fx + Fx'$ is in $\mathcal{P}_2$, and then $\mathcal{L}(Fx) \cap \mathcal{L}(Fx') \supseteq \mathcal{L}(P)$ since $\mathcal{L}$ inverts inclusion, so $\mathcal{L}(Fx) = C \star x$ and $\mathcal{L}(Fx') = C \star x'$ are connected. The equality follows from Proposition 5.3.(2).

The left spaces $C \star x$ and $C \star x'$ are connected when $C \star x \cap C \star x' = P \in \mathcal{P}_2$, and then $\mathcal{RR}(C \star x) \cap \mathcal{RR}(C \star x') \supseteq \mathcal{RR}(P) = \mathcal{L}(P) \in \mathcal{P}_2$, so $x \star C = \mathcal{RR}(C \star x)$ and $x' \star C = \mathcal{RR}(C \star x')$ are connected.

The right spaces $x \star C$ and $x' \star C$ are connected when $x \star C \cap x' \star C = P \in \mathcal{P}_2$, and then $\mathcal{L}(x \star C) + \mathcal{L}(x' \star C) \subseteq \mathcal{L}(P) \in \mathcal{P}_2$, so $Fx = \mathcal{L}(x \star C)$ and $Fx' = \mathcal{L}(x' \star C)$ are connected. □

Remark 6.4. Theorem 6.2 follows immediately from Proposition 6.3 and the fact that $\rho\rho\rho$ is the identity. Indeed, if $P \in \mathcal{P}_2$ is written as $P = Fx + Fx'$, then $\mathcal{LLL}(P) = \mathcal{LL}(\rho(Fx) + \rho(Fx')) = \mathcal{L}(\rho\rho(Fx) \cap \rho\rho(Fx')) = \rho\rho\rho(Fx) + \rho\rho\rho(Fx') = Fx + Fx' = P$.

Theorem 6.5. *The triality map ρ is an automorphism of the graph defined on $\mathcal{P}_1 \cup \mathcal{P}_4 \cup \mathcal{P}_{4'}$.*

Proof. There are six statements to verify, one for every pair of classes from $\mathcal{P}_1$, $\mathcal{P}_4$ and $\mathcal{P}_{4'}$. The cases where the vertices belong to the same class, are in Proposition 6.3.

(1) Fx is incident with $C \star x'$ when $Fx \subseteq C \star x' = \mathrm{Ker}(L_{x'})$. Then $x' \star x = 0$ so $\mathcal{L}(Fx) \cap \mathcal{RR}(C \star x') = C \star x \cap x' \star C \supseteq x' \star x^{\perp} \in \mathcal{P}_3$.
(2) $C \star x$ and $x' \star C$ are incident when $x' \star x = 0$, so $x' \in \mathrm{Ker}(R_x) = \mathrm{Im}(L_x)$; thus $\mathcal{L}(x' \star C) = Fx' \subseteq x \star C = \mathcal{RR}(C \star x)$.
(3) Fx is incident with $x' \star C$ when $Fx \subseteq x' \star C$, in which case $C \star x = \mathcal{L}(Fx) \supseteq \mathcal{L}(x' \star C) = Fx'$. □

7. Linkage

We prove that the diameter of the graph defined in Sect. 6.2 is at most three. First we show that the graph can be reconstructed from its natural 3-partite subgraph:

Proposition 7.1. *The following are equivalent for two left maximal isotropic spaces $U, U' \in \mathcal{P}_4$:*

(1) *U and U' are connected by an edge in $\mathcal{P}_2$;*
(2) *there is some $W \in \mathcal{P}_{4'}$ connected to both U and U';*
(3) *there is some $K \in \mathcal{P}_1$ connected to both U and U'.*

Proof. (1) $\Longleftrightarrow$ (3): U and U' are connected iff $U \cap U' \neq 0$ iff there is some $K \in \mathcal{P}_1$ such that $K \subseteq U \cap U'$.

(1) $\Longrightarrow$ (2): Write $U = C \star x$ and $U' = C \star x'$. By definition U and U' are connected iff $\mathrm{B}(x, x') = 0$, in which case $P = Fx + Fx' \in \mathcal{P}_2$, and for every $y \in \mathcal{R}(P)$, $y \star C$ is connected to U and U' by Proposition 3.8 since $y \star x = y \star x' = 0$.

(2) $\Longrightarrow$ (1): If $\dim(U \cap W), \dim(U' \cap W) = 3$, then $\dim(U \cap U') \geq \dim((U \cap W) \cap (U' \cap W)) = 2$ so U and U' are connected. □

Proposition 7.2. *The distance between any $U \in \mathcal{P}_4$ and any $U' \in \mathcal{P}_{4'}$ is at most* 3. *In other words, for some $W \in \mathcal{P}_4$ and $W' \in \mathcal{P}_{4'}$, the intersections $U \cap W'$, $W' \cap W$ and $W \cap U'$ are all* 3*-D.*

Proof. If $\dim(U \cap U') = 3$ then take $W = U$ and $W' = U'$. Otherwise, by Proposition 3.8, $U \cap U'$ is 1-D. Let P be any 2-D subspace of U containing $U \cap U'$. Let $P' \subseteq P^{\perp} \cap U'$ be a 2-D space. Then $S = P + P' \in \mathcal{P}_3$, so by Proposition 4.2 there are unique spaces $W \in \mathcal{P}_4$ and $W' \in \mathcal{P}_{4'}$ such that $W \cap W' = S$. But then $P \subseteq U \cap W'$ and $P' \subseteq W \cap U'$, where in each case the spaces are of different kinds. Therefore $\dim(U \cap W'), \dim(W \cap U') = 3$. □

Proposition 7.3. *The diameter of the graph $\mathcal{P}_4$ is* 2. *In other words, for every $U, W \in \mathcal{P}_4$, there is some $V \in \mathcal{P}_4$ for which* $\dim(U \cap V) = 2$ *and* $\dim(W \cap V) = 2$.

Proof. Writing $U = C \star x$ and $W = C \star y$, the condition on $V = C \star z'$ is that z be orthogonal to x and y, so take any $0 \neq z' \in C \star x \cap (Fx + Fy)^{\perp}$. One could also take an arbitrary $U' \in \mathcal{P}_{4'}$ connected to W, and apply Propositions 7.2 and 7.1. □

The analog of Proposition 7.3 for $\mathcal{P}_1$, namely that for every $K, K' \in \mathcal{P}_1$ there is some $K'' \in \mathcal{P}_1$ such that $K + K''$ and $K' + K''$ are in $\mathcal{P}_2$, can be proved directly by a similar argument: intersect $(K + K')^{\perp}$, which is 6-D, with any 4-D isotropic subspace.

Theorem 7.4. *The distance between two vertices in a class is at most* 2, *and between vertices in different classes is at most* 3. *In particular the diameter of the graph on $\mathcal{P}_1 \cup \mathcal{P}_4 \cup \mathcal{P}_{4'}$ is* 3.

Proof. Apply a suitable power of ρ to Propositions 7.2 and 7.3. □

8. strongly orthogonal pairs

We say that non-zero isotropic elements $x, y \in C$ are **strongly orthogonal** if $\mathrm{B}(x, y) = 0$ and $x \star y = 0$. Recall that by Remark 3.3, the second condition alone implies that x and y are isotropic. When this is the case, we denote $x \rightsquigarrow y$, or $y \leftsquigarrow x$; the relation is not symmetric. Note that x and y are connected in $\mathcal{P}_1$ via the graph defined in Sect. 6 iff $\mathrm{B}(x, y) = 0$, so the new relation is a finer one. Since the conditions are independent of scalars, we may write $Fx \rightsquigarrow Fy$ instead of $x \rightsquigarrow y$.

8.1. The square of an isotropic element

Remark 8.1. If $x \star y = 0$ then $(C \star y) \star x = Fy$ and $y \star (x \star C) = Fx$ by Equations (2.3)–(2.4).

Proposition 8.2. *Let x, x' be isotropic vectors.*

(1) *$z = x' \star x$ solves the equation $x \star z = z \star x' = 0$.*
(2) *If $x' \star x \neq 0$, then $z = x' \star x$ is the only solution to $x \star z = z \star x' = 0$, up to scalar multiple.*

(3) *If* $x \star x \neq 0$ *then the only solution to* $x \star z = z \star x = 0$ *is, up to scalar multiples,* $z = x \star x$.

Proof. (1) By (2.2), $x \star (x' \star x) = (x' \star x) \star x' = 0$.
(2) If $x \star z = z \star x' = 0$ then $z \in \mathrm{Ker}(L_x) \cap \mathrm{Ker}(R_{x'}) = \mathrm{Im}(R_x) \cap \mathrm{Im}(L_{x'}) = F \cdot (x' \star x)$ by Proposition 3.8.(1).
(3) This is a special case of (2).

□

Symmetric composition algebras in general are not power-associative. However since $x \star (x \star x) = (x \star x) \star x = 0$ for any isotropic element, $x^3 = 0$ in any interpretation of the left-hand side. We let x^4 denote the only product in which x participates four times which is not identically zero, namely $x^4 = (x \star x) \star (x \star x)$.

Remark 8.3. $x \star x = 0$ iff $C \star x \subseteq x^{\perp}$ iff $x \star C \subseteq x^{\perp}$, since $\mathrm{B}(C \star x, x) = \mathrm{B}(C, x \star x)$.

Notice that by (2.5), $x^4 = \mathrm{B}(x \star x, x)x \in Fx$ for every isotropic element x. By Proposition 8.2.(3), if $x \star x \neq 0$, the only potential completion of the chain $x \rightsquigarrow * \rightsquigarrow x$ is by plugging $x \star x$ in the middle. Indeed $x \rightsquigarrow x \star x \rightsquigarrow x$ iff $x^4 = 0$, since this is when $\mathrm{B}(x, x \star x) = 0$.

8.2. *Chains of strongly orthogonal elements*

Proposition 8.4. (1) *If* $x \rightsquigarrow y$ *and* $x \star x \neq 0$, *then* $y \rightsquigarrow x \star x$.
(2) *Assuming* $x^4 \neq 0$ *and* $y^4 \neq 0$, *each of the following implies the others:* $x \rightsquigarrow y$, $y \rightsquigarrow x \star x$, $x \star x \rightsquigarrow y \star y$, $y \star y \rightsquigarrow x$.

Proof. (1) $\mathrm{B}(y, x \star x) = \mathrm{B}(x \star x, y) = \mathrm{B}(x, x \star y) = 0$, and $y \star (x \star x) = -x \star (x \star y) = 0$ by the relation (2.4).
(2) Each relation implies the one following it by (1), and from $y \star y \rightsquigarrow x$ we get $x \rightsquigarrow y^4 \equiv y$.

□

We say that C is **reduced** if $x \star x \neq 0$ for every $x \neq 0$.

Proposition 8.5. *Suppose* C *is reduced. If the chain of length* n, $x_1 \rightsquigarrow * \rightsquigarrow \cdots \rightsquigarrow * \rightsquigarrow x_n$, *can be completed for every* $x_1, x_n \in \mathcal{P}_1$, *then the same holds for every chain of the same length, regardless of the direction of the arrows.*

Proof. By Proposition 8.4.(2), $u \rightsquigarrow v \rightsquigarrow w$ iff $u \leftsquigarrow v \star v \leftsquigarrow w$, so replacing an entry by its square changes the order of all the arrows incoming to or outgoing from this entry. We may also change x_1 to $x_1 \star x_1$ or x_n to $x_n \star x_n$, so there are sufficiently many inversions to make all the chains of a given length equivalent. □

We now prove that the diameter of $\mathcal{P}_1$, as a directed graph with respect to the strong orthogonality relation $\rightsquigarrow$, is at most 4. We need one preparatory remark.

Proposition 8.6. (1) *For any $w, w', w \rightsquigarrow w'$ iff $w \in (w' \star C) \cap w'^{\perp}$, iff $w' \in (C \star w) \cap w^{\perp}$.*
(2) *If $y' \star x' = 0$ then the chain $x' \rightsquigarrow * \rightsquigarrow y'$ can be completed.*

Proof. (1) This is because $\mathrm{Ker}(L_w) = \mathrm{Im}(R_w)$ and $\mathrm{Ker}(R_w) = \mathrm{Im}(L_w)$.
(2) The condition for $x' \rightsquigarrow z \rightsquigarrow y'$ is that $z \in (C \star x') \cap (y' \star C) \cap x'^{\perp} \cap y'^{\perp}$, and by assumption this is non-zero since $(C \star x') \cap (y' \star C)$ is 3-D by Proposition 3.8. □

Remark 8.7. From Proposition 8.6.(2) it follows that every arrow $y' \rightsquigarrow x'$ can be completed to a triangle $z' \rightsquigarrow y' \rightsquigarrow x' \rightsquigarrow z'$.

Since the graph is directed, we have several variants of the main result. Of course when C is reduced they are all equivalent by Proposition 8.5.

Theorem 8.8. *Let x, y be non-zero isotropic elements. Then one can complete each of the chains*

$$x \rightsquigarrow * \rightsquigarrow * \rightsquigarrow * \rightsquigarrow y,$$

$$x \leftsquigarrow * \rightsquigarrow * \rightsquigarrow * \rightsquigarrow y,$$

and

$$x \rightsquigarrow * \rightsquigarrow * \rightsquigarrow * \leftsquigarrow y,$$

with isotropic non-zero elements.

Proof. We repeatedly use Proposition 8.6.(1). The proof in each case begins with (2) of that Proposition: we find x' and y' with $y' \star x' = 0$, which will guarantee the chain can be completed.

1. For $x \rightsquigarrow x'$ and $y' \rightsquigarrow y$: Choose $x' \in (C \star x) \cap x^{\perp}$ satisfying $\mathrm{B}(y, x') = 0$ as well; then $(y \star C) \cap (x' \star C)$ has dimension 2 by Proposition 3.7, so there is a non-zero $y' \in (y \star C) \cap y^{\perp} \cap (x' \star C)$. Thus $y' \rightsquigarrow y$ and $y' \star x' = 0$.

2. For $x \leftsquigarrow x'$ and $y' \rightsquigarrow y$: Choose $x' \in (x \star C) \cap x^{\perp}$ with $\mathrm{B}(y, x') = 0$; then $(y \star C) \cap (x' \star C)$ has dimension 2, so there is a non-zero $y' \in (y \star C) \cap y^{\perp} \cap (x' \star C)$, and then $y' \rightsquigarrow y$ and $y' \star x' = 0$ as above.

3. For $x \rightsquigarrow x'$ and $y' \leftsquigarrow y$: Choosing x' orthogonal to y will not insure that $\dim((C \star y) \cap (x' \star C)) > 1$, so instead we choose $y' \in C \star y \cap y^{\perp}$, which is also orthogonal to x. Then $\dim((C \star y') \cap (C \star x)) = 2$, and we can find a non-zero x' such that $x \rightsquigarrow x'$ and $y' \star x' = 0$, as required. □

Remark 8.9. The arguments in Theorem 8.8 do not apply if the direction of the arrows in the chain is not one of the above. For example, when C is not reduced, it is not clear if every chain of the form

$$x \leftsquigarrow * \rightsquigarrow * \rightsquigarrow * \leftsquigarrow y$$

or

$$x \rightsquigarrow * \leftsquigarrow * \rightsquigarrow * \leftsquigarrow y$$

can be completed.

Theorem 8.8, in particular in the form $x \rightsquigarrow * \leftrightsquigarrow * \rightsquigarrow * \leftrightsquigarrow y$, which follows in C is reduced, generalizes the common slot lemma for Kummer elements in central simple algebras of degree 3 due to Rost [10]; see Theorem 13.2 below.

9. *p*-central spaces

The rest of this article is devoted to applications to central simple algebras of degree $p = 3$. A non-central element of a central simple algebra is called p-central if $\alpha = x^p \in F$. When the base field contains a p-root of unity $\rho \in F$, every p-central element can be complemented by an element y satisfying $yxy^{-1} = \rho x$; then y is necessarily p-central, and $A = F[x, y]$ is fully characterized by $\alpha = x^p$ and $\beta = y^p$. Such an algebra is called a symbol algebra, denoted by $(\alpha, \beta)_p$. The generators x and y form a standard pair of generators for A. In order to understand the various presentations of A in these terms, we must understand the p-central elements. Here we are mostly concerned with p-central subspaces, which are linear subspaces all of whose elements are p-central.

In an algebra of degree 3, every element a satisfies the characteristic polynomial

$$P_a(\lambda) = \lambda^3 - s_1(a)\lambda^2 + s_2(a)\lambda - s_3(a),$$

where $s_i : A \to F$ are the reduced coefficients: s_1 is the reduced trace, and s_3 is the reduced norm. Thus, the 3-central elements in A are the isotropic vectors of the quadratic form s_2 restricted to the space of zero-trace elements. In particular, a p-central space is an isotropic subspace.

Let x be a 3-central element, so conjugation by x induces an automorphism of order p of A. Let V_j be the eigenspace of the eigenvalue ρ^{-j}. Thus $V_0(x) = F[x]$, and that every $V_j(x)$ is an $F[x]$-module. In fact $V_j(x) = F[x]y^j$, where $y \in A^\times$ is any element such that $yxy^{-1} = \rho x$. Note that $V_2(x) = V_1(x^2)$.

Proposition 9.1. *Let x be a p-central element, and let k be prime to p. Then*

$$V = Fx + V_k(x)$$

is a p-central space of A, maximal with respect to inclusion.

Proof. It suffices to prove the claim for $k = 1$. Taking $y \in A$ such that $yxy^{-1} = \rho x$, we have that $x+y$ is p-central, and since x and wy form a standard pair of generators for every $w \in F[x]$, the elements $x + wy$ are all p-central. The maximality follows since the dimension of an isotropic subspace is at most half the dimension of the space of zero-trace elements, which is 8.

(The claim is true, with a different proof, for any prime p). □

As for the uniqueness of presentation, we have:

Proposition 9.2. *Assume $p \neq 2$, and let x, y be a standard pair of generators.*

(1) *Let $\hat{x}, \hat{y} \in V = Fx + F[x]y$ be non-zero elements satisfying $\hat{y}\hat{x} = \rho\hat{x}\hat{y}$. Then $\hat{x} \in Fx + Fx\hat{y}$ and $F[x]\hat{y} = F[x]y$.*

(2) *If, furthermore,* $\hat{x}\hat{y} \in V$*, then* $F\hat{x} = Fx$.

Proof. Write $\hat{x} = \alpha x + wy$ and $\hat{y} = \beta x + w'y$ for $\alpha, \beta \in F$ and $w, w' \in F[x]$. Let σ be the automorphism of $F[x]$ induced by conjugation by y. The condition $\hat{y}\hat{x} = \rho\hat{x}\hat{y}$ gives

$$\begin{aligned}\alpha\beta x^2 &+ (\beta x w + \rho\alpha x w')y + w'\sigma(w)y^2 \\ &= \rho\alpha\beta x^2 + (\rho^2\beta x w + \rho\alpha x w')y + \rho w\sigma(w')y^2,\end{aligned}$$

which implies $\alpha\beta = 0$. If $\beta \neq 0$ then $\alpha = 0$ implies $w = 0$, which is impossible. Therefore $\beta = 0$, and the remaining equation is $w'\sigma(w) = \rho w\sigma(w')$, from which it follows that $w \in Fxw'$ and $F[x]\hat{y} = F[x]y$.

The condition $\hat{x}\hat{y} \in V$ then implies $w = 0$ or $w' = 0$, but $w' \neq 0$, so $\hat{x} = \alpha x$. □

Corollary 9.3. $Fx + V_j(x) = F\hat{x} + V_j(\hat{x})$ *iff* $F\hat{x} = Fx$.

Indeed, the proposition covers $j = 1$, and replacing ρ by ρ^j gives the general case.

10. The Okubo algebra

Let A be a cyclic central simple algebra of degree 3 over F. We assume from here on that char $F \neq 3$, and that F contains a 3-root of unity, ρ. Since 3-central elements have trace zero, we focus on the space $C = \{a \in A : \operatorname{Trd}(a) = 0\}$ of elements with trace zero in A; as vector spaces, $A = F \oplus C$.

But C is not a subalgebra of A, so one defines an operation on C, called the **Okubo product** [6, Sect. 34.C], by setting

$$a \star b = \frac{1-\rho}{3}ab + \frac{1-\rho^2}{3}ba - \frac{1}{3}\operatorname{Trd}(ab), \tag{10.1}$$

where ab is the usual product in A. Under this action, C is a symmetric composition algebra with norm

$$\mathrm{N}(a) = -\frac{1}{3}s_2(a), \tag{10.2}$$

where s_2 is the second characteristic coefficient [6, Proposition 34.19]. (Operations of the form $a * b = \lambda ab + (1-\lambda)ba$ for a fixed $\lambda \in F$ are called quasi-associative, see [11, Sect. V.3]).

Note that when A is a division algebra, $(C, \star)$ has no idempotents: $a \star a = a^2 - \frac{1}{3}\operatorname{Trd}(a^2)$ cannot be equal to a since A has no quadratic subfields. For the same reason $(C, \star)$ is reduced, namely $a \star a \neq 0$ unless $a = 0$. We use a^2 to denote the usual product aa in A; the square in C will always be written explicitly, as $a \star a$. Likewise a^{-1} will always denote the inverse in A.

As noted above, $\mathrm{N}(a) = 0$ if and only if the characteristic polynomial is $P_a(\lambda) = \lambda^3 - s_3(a)$, which is the case exactly when a is 3-central.

We can now compute the left and right maximal isotropic spaces.

Remark 10.1. Let x be a 3-central element. Then

$$x \star C = Fx^2 + V_2(x),$$

and

$$C \star x = Fx^2 + V_1(x).$$

Proof. Let y be an element such that $yxy^{-1} = \rho x$. Since $\frac{1-\rho}{3}\rho^2 + \frac{1-\rho^2}{3} = 0$, we have that $x \star F[x]y = F[x]y^2 \star x = 0$. Likewise computation shows that $x \star F[x]y^2 = F[x]y^2$ and $F[x]y \star x = F[x]y$. We also have that $x \star (Fx + Fx^2) = (Fx + Fx^2) \star x = Fx^2$ (notice that $x \star F$ is not defined, as $F \not\subseteq C$). □

Recall that every central simple algebra of degree 3 is cyclic [1, Theorem XI.5]. Our main result in this section complements Theorem 9.1 as follows.

Theorem 10.2. *Let A be a cyclic algebra of degree 3 over a field F containing primitive 3-roots of unity. Then every maximal 3-central subspace of A has the form $Fx + V_1(x)$ or $Fx + V_2(x)$ for some 3-central element x.*

Proof. Let $V \subseteq A$ be a maximal 3-central subspace. Then $V \subseteq C$ where $C = A_0$ is the space of elements with trace zero, which is a composition algebra under the Okubo product (10.1), with respect to the norm (10.2). By Theorem 3.1 and Proposition 3.2, V has the form $V = x \star C$ or $V = C \star x$ for a 3-central x. Now let $y \in A$ be an element such that $yxy^{-1} = \rho x$. By Remark 10.1, $x \star C = Fx^2 + V_2(x)$ and $C \star x = Fx^2 + V_1(x)$, as claimed. □

As for the intersections of such spaces, we have:

Proposition 10.3. *Let x and x' be 3-central elements, and $k = 1, 2$.*

(1) $(Fx + V_k(x)) \cap (Fx' + V_k(x'))$ *is 2-D if* $\mathrm{Trd}((xx')^{-1}) = 0$, *and zero otherwise.*
(2) $(Fx + V_1(x)) \cap (Fx' + V_2(x'))$ *is 3-D if* $x^2 \star x'^2 = 0$, *and equals* $F \cdot (x^2 \star x'^2)$ *otherwise.*

This is immediate from Propositions 3.7 and 3.8, using $C \star x^2 = Fx + V_2(x)$ and $x^2 \star C = Fx + V_1(x)$.

11. Explicit presentation

Fix a standard pair of generators x, y for the algebra A, and let $\alpha = x^3$ and $\beta = y^3$ be the defining scalars. We record, for later use, some basic computations in the Okubo algebra C associated to A. Note that C is spanned by $\{x^i y^j\}$ where $(i, j) \in \{0, 1, 2\}^2$ range over the non-zero vectors. If the product of distinct basis elements is not a scalar, then their (multiplicative) commutator is ρ or ρ^2.

11.1. The operation $\star$

Directly from the definition (10.1) we obtain

$$\begin{aligned} x \star x &= x^2; \\ x \star x^{-1} &= 0; \\ x \star y &= 0; \\ y \star x &= -\rho^2 xy. \end{aligned} \tag{11.1}$$

These four cases encode every product of the form $x^i y^j \star x^{i'} y^{j'}$; for convenience, here is the complete multiplication table (with column element acting from the left).

$\star$	x	x^2	y	xy	x^2y	y^2	xy^2	x^2y^2
x	x^2	0	0	0	0	$-\rho xy^2$	$-\rho x^2y^2$	$-\rho\alpha y^2$
x^2	0	αx	$-\rho x^2 y$	$-\rho\alpha y$	$-\rho\alpha xy$	0	0	0
y	$-\rho^2 xy$	0	y^2	$-\rho^2 xy^2$	0	0	$-\rho^2\beta x$	0
xy	$-\rho^2 x^2 y$	0	0	$\rho x^2 y^2$	$-\alpha y^2$	$-\rho\beta x$	0	0
x^2y	$-\rho^2\alpha y$	0	$-\rho x^2 y^2$	0	$\rho^2\alpha xy^2$	0	0	$-\alpha\beta x$
y^2	0	$-\rho^2 x^2 y^2$	0	0	$-\rho^2\beta x^2$	βy	0	$-\rho^2\beta x^2 y$
xy^2	0	$-\rho^2\alpha y^2$	0	$-\beta x^2$	0	$-\rho\beta xy$	$\rho^2\beta x^2 y$	0
x^2y^2	0	$-\rho^2\alpha xy^2$	$-\rho\beta x^2$	0	0	0	$-\alpha\beta y$	$\rho\alpha\beta xy$

11.2. The bilinear form

The (symmetric) bilinear form associated to N is defined by $\mathrm{B}(a, b) = \mathrm{N}(a+b) - \mathrm{N}(a) - \mathrm{N}(b)$. Linearization of $\mathrm{B}(a, a) = \frac{1}{3}\,\mathrm{Trd}(a^2)$ gives

$$\mathrm{B}(a, b) = \frac{1}{3}\,\mathrm{Trd}(ab). \tag{11.2}$$

It follows that

$$\begin{aligned} \mathrm{B}(x, x) &= 0; \\ \mathrm{B}(x, x^{-1}) &= 1; \\ \mathrm{B}(x, y) &= 0; \end{aligned} \tag{11.3}$$

This can be verified directly, as $\mathrm{N}(x + x^{-1}) = 1$ and $\mathrm{N}(x + y) = 0$.

Using the Okubo product (10.1), we can easily identify standard pairs of generators. Indeed, the product can be rewritten as

$$a \star b = \frac{1-\rho^2}{3}(ba - \rho ab) - \frac{1}{3}\,\mathrm{Trd}(ab),$$

which emphasizes its connection to the relation $ba = \rho ab$.

11.3. Unital translation

Since y is 3-central, we have that $(y+y^{-1})^3 - 3(y+y^{-1}) = (y^3+y^{-3}) \in F$, so $s_2(y+y^{-1}) = -3$ and $\mathrm{N}(y+y^{-1}) = 1$. Thus we may choose $u = y+y^{-1}$ in (2.9), and then $e = (y+y^{-1}) \star (y+y^{-1}) = y^2+y^{-2}$ is the unit element of $(C, \diamond)$.

11.4. The involution

A unital composition algebra $(C, \diamond, e)$ has an involution

$$a \mapsto \bar{a} \tag{11.4}$$

defined by $a + \bar{a} = \mathrm{B}(a, e)e$.

Since $e = y^2 + y^{-2}$, one computes directly using (11.3) that

$$\overline{x} = -x$$

and

$$\overline{y} = \beta^2 y^{-1}.$$

11.5. The operation $\diamond$

With $u = y + y^{-1}$ in (2.9), we have

$$a \diamond b = ((y+y^{-1}) \star a) \star (b \star (y+y^{-1})). \tag{11.5}$$

Since (C, N) has isotropic vectors, it is known, a-priori, that $(C, \diamond)$ is the split octonion algebra. However it turns out that for *any* standard pair of generators x, y, one can scale the standard monomial basis $\left\{x^i y^j\right\}$ of C, so that the multiplication table of $\diamond$ is fixed monomial table, independent of the original algebra A, or of the choice of x and y. Indeed, with respect to the basis

$$B = \Big\{y^2, y^{-2}, x, -x^{-1}, v_1 = xy^{-2}, v_2 = -\rho^2 x^2 y^{-2}, \\ v_3 = x^{-2}y^2, v_4 = -\rho x^{-1} y^2\Big\},$$

the multiplication table is (with the row entry acting from the left, and zero entries omitted):

$\diamond$	y^2	y^{-2}	x	$-x^{-1}$	v_1	v_2	v_3	v_4
y^2	y^2		x		v_1		v_3	
y^{-2}		y^{-2}		$-x^{-1}$		v_2		v_4
x		x		y^2	v_2		$-v_4$	
$-x^{-1}$	$-x^{-1}$		y^{-2}			v_1		$-v_3$
v_1		v_1	$-v_2$				$-x^{-1}$	y^2
v_2	v_2			$-v_1$			y^{-2}	x
v_3		v_3	v_4		$-x^{-1}$	y^2		
v_4	v_4			v_3	y^{-2}	x		

Evidently, $e_{11} = y^2, e_{22} = y^{-2}, e_{12} = x$ and $e_{21} = -x^{-1}$ form a system of matrix units, and $i = y^2 - y^{-2}$ and $j = x - x^{-1}$ give the quaternionic presentation

$$i \diamond i = j \diamond j = e, \qquad j \diamond i = -i \diamond j$$

of a 2×2 matrix subalgebra of C.

Since the vectors x, x^{-1} and the v_i are all orthogonal to $y^2 + y^{-2}$ with respect to B, the involution described in Sect. 11.4 satisfies

$$\overline{y^2} = y^{-2}, \qquad \overline{y^{-2}} = y^2, \qquad \overline{x} = -x, \qquad \overline{x^{-1}} = -x^{-1}, \qquad \overline{v_i} = -v_i.$$

In particular $\overline{i} = -i$ and $\overline{j} = -j$, so (11.4) is the symplectic involution on the subalgebra $F[i, j] = Fe + Fi + Fj + F(i \diamond j)$.

Let $k = v_1 + v_4$. Then $k \diamond k = e$, $\overline{k} = -k$, and for every $a, b \in F[i, j]$ we have

$$\begin{aligned} a \diamond (k \diamond b) &= k \diamond (\overline{a} \diamond b), \\ (k \diamond b) \diamond a &= k \diamond (a \diamond b), \\ (k \diamond a) \diamond (k \diamond b) &= \quad b \diamond \overline{a}. \end{aligned} \tag{11.6}$$

This completes the description of $(C, \diamond)$ as a split Cayley-Dickson extension $(\mathrm{M}_2(F), 1)$.

12. Standard pairs of generators in isotropic spaces

Again let A be cyclic central simple algebra of degree 3 over F. Consider the variety $\mathrm{XY}_A \subseteq \mathbb{P}A \times \mathbb{P}A$ of pairs (x, y) for which $yx = \rho xy$. Every point in XY_A corresponds to a standard pair of generators.

In fact $\mathrm{XY}_A \subseteq \mathbb{P}C \times \mathbb{P}C$. The projection on each entry covers the variety of 3-central elements, and the fibers are all 3-D subspaces: the fiber over x is $V_1(x) = F[x]y$ where $yxy^{-1} = \rho x$. In this section we consider standard pairs of generators inside a given 3-central space $U \subseteq C$, by computing the intersection of XY_A with $\mathbb{P}U \times \mathbb{P}U$.

Now we have:

Proposition 12.1. (1) *For a* 3*-central element* z, $z \star w = 0$ *iff* $w \in Fz^2 + V_1(z)$.
(2) *Two* 3*-central elements* $z, w \in A$ *form a standard pair of generators iff* $\mathrm{B}(z, w) = 0$ *and* $z \star w = 0$.

Proof. (1) Follows from $\mathrm{Ker}(L_z) = \mathrm{Im}(R_z)$ (Proposition 3.4), where $L_z : x \mapsto z \star x$ and $R_z : x \mapsto x \star z$ are the multiplication operators. A computational verification runs as follows. From (11.4) we see that $z \star w = 0$ implies $wz - \rho zw \in F$, and for a suitable $\theta \in F$, $y = w - \theta z^{-1}$ satisfies $yz = \rho zy$. The other direction follows from (11.1) and (11.3).
(2) Immediate from the first statement, and the fact that when $u = w + \theta z^{-1}$ and $wzw^{-1} = \rho z$, $\theta = \mathrm{B}(z, u)$. ◻

We write $a \rightsquigarrow b$ ('a is **strongly orthogonal** to b'), if $a \star b = 0$ and $\mathrm{B}(a, b) = 0$ (this is not symmetric). Thus $a \rightsquigarrow b$ iff a and b form a standard pair of generators.

12.1. Maximal subspaces

From Theorem 10.2 it is clear that every maximal isotropic subspace contains a standard pair of generators.

Moreover, by Corollary 9.3, a maximal subspace uniquely determines its generator x (up to multiplication by scalars), which provides an inner characterization of the kind: in a right maximal 3-central space there is an element y' such that $y'xy'^{-1} = \rho x$, and in a left space there is an element such that $y'xy'^{-1} = \rho^{-1}x$ (but it is impossible to have both).

Following Proposition 9.2.(1), the standard pairs of generators in a maximal 3-central space can be described as follows:

Proposition 12.2. *Let V be a maximal 3-central space. Then $\mathrm{XY}_A \cap (\mathbb{P}V \times \mathbb{P}V)$ is a blowup of $\mathbb{P}V$ at a point. More precisely:*

(1) *Suppose V is of the right kind, namely $V = Fx \oplus W$ for $W = V_1(x)$, where x is a 3-central element. Then*

$$\mathrm{XY}_A \cap (\mathbb{P}V \times \mathbb{P}V) = \cup_{y\in\mathbb{P}W}(\mathbb{P}(Fx + Fxy) \times \{y\}).$$

(2) *Suppose V is of the left kind, namely $V = Fx \oplus W$ for $W = V_2(x)$. Then*

$$\mathrm{XY}_A \cap (\mathbb{P}V \times \mathbb{P}V) = \cup_{y\in\mathbb{P}W}(\{y\} \times \mathbb{P}(Fx + Fxy)).$$

12.2. 3-D spaces

We now describe the 3-D spaces of 3-central elements. As mentioned in the introduction, Raczek [9] obtained a similar classification by more direct means.

Theorem 12.3. *Every 3-D isotropic space in A is of the form*

$$U = Fy + Fxy + Fz \tag{12.1}$$

for a (projectively) unique standard pair of generators x, y and a unique non-zero $Fz \subseteq Fx + Fx^2y$.

Proof. By Proposition 4.2, every such space can be (projectively) uniquely presented in the form $U = x'' \star x'^{\perp} = x'' \star C \cap C \star x'$, where x'', x' are isotropic vectors satisfying $x'' \star x' = 0$. Since $Fx'' = Fx''^4$, by changing x'' up to scalars we may assume $x'' = x^2$ for some isotropic element x. By Proposition 12.1, the condition $x^2 \star x' = 0$ implies either $x' \in Fx$ or $x' = y^{-1} + \rho^{-1}\theta x$ for some element y satisfying $yxy^{-1} = \rho x$ and some $\theta \in F$.

For the first case we have $x^2 \star C = Fx + F[x]y$ and $C \star x = Fx^2 + F[x]y$, so the intersection is $F[x]y$, in which case $z = x^2y$.

In the second case, let $y' = xy$, which together with $x' = y^{-1} + \rho^{-1}\theta x$ forms a standard pair of generators. Then

$$x^2 \star C \cap C \star x' = (x^2 \star C) \cap \mathrm{Ker}(L_{x'})$$

is equal to $Fy + Fxy + F(x + \theta x^2y)$ since $x' \star y = x' \star xy = x' \star (x + \theta x^2y) = 0$. So take $z = x + \theta x^2y$ (then $x' = zx^{-1}y^{-1}$). □

We can now compute $\mathrm{XY}_A \cap (\mathbb{P}U \times \mathbb{P}U)$.

Proposition 12.4. *Let U be a 3-D 3-central space, presented as in (12.1), with $z = \varphi x + \psi x^2 y$ for some $\varphi, \psi \in F$, and let $\alpha = x^3$. The intersection $\mathrm{XY}_A \cap (\mathbb{P}U \times \mathbb{P}U)$ is equal to:*

(1) *The plane $\left\{(x\hat{y}, \hat{y}) : \hat{y} \in \mathbb{P}U\right\}$ if $Fz = Fx^2y$;*
(2) *The union of intersecting lines*

$$\{[x]\} \times \mathbb{P}(Fy + Fxy) \cup \mathbb{P}(Fx + Fxy) \times \{[y]\}$$

if $Fz = Fx$;
(3) *Otherwise, the intersection is the line*

$$\left\{([\alpha\psi\lambda xy + \mu z], [\lambda y + \mu xy]) : (\lambda, \mu) \in F^2 - \{(0, 0)\}\right\}.$$

Proof. Suppose $\hat{x}, \hat{y} \in U$ satisfy $\hat{y}\hat{x} = \rho\hat{x}\hat{y}$. By Proposition 12.2, $\hat{y} \in F[x]y \cap U = Fy + Fxy$, and $\hat{x} \in U \cap (Fx + Fx\hat{y})$. Write $\hat{y} = \lambda y + \mu xy$ for some $\lambda, \mu \in F$. If $\varphi = 0$ (this is Case (1)) then $U = F[x]y$ and the claim follows by direct computation. So we may assume $\varphi \neq 0$.

The intersection $U \cap (Fx + Fx\hat{y})$ is the 1-D space spanned by $\mu\varphi x + \psi x\hat{y} = \psi\lambda xy + \mu z$, unless $Fz = Fx$ and $F\hat{y} = Fy$, where the intersection is the 2-D space $Fx + Fxy$. □

Having classified the arrows $x \rightsquigarrow y$ in $\mathbb{P}U$, we classify all chains $\hat{x} \rightsquigarrow \hat{y} \rightsquigarrow \hat{z}$.

Proposition 12.5. *Let U be a 3-D 3-central space, presented as in (12.1).*

(1) *When $Fz = Fx^2y$, U is the eigenspace $V_1(x)$, and the triplets $\hat{x} \rightsquigarrow \hat{y} \rightsquigarrow \hat{z}$ all have the form $\hat{y} \rightsquigarrow x^2\hat{y} \rightsquigarrow x\hat{y}$ for $\hat{y} \in U$. Every such triplet closes into a triangle: $x\hat{y} \rightsquigarrow \hat{y}$.*
(2) *Otherwise, the (projectively) unique triplet in U is*

$$z \rightsquigarrow xy \rightsquigarrow y;$$

Proof. The relations $\hat{x} \rightsquigarrow \hat{y}$ and $\hat{y} \rightsquigarrow \hat{z}$ in U are classified in Proposition 12.4, so it remains to compare left terms to right terms, checking the case $Fz = Fx$ separately. □

12.3. 2-D 3-central spaces

Following the definition of the operator $\mathcal{L}$ in (5.1), we define for a 3-central space U

$$\mathcal{L}(U) = \cap_{x \in U}(Fx^2 + V_1(x));$$

namely, the intersection of the left maximal 3-central subspaces associated to elements of U.

Theorem 12.6. *For any* 2*-D* 3*-central space* P*, we have the following:*

(1) $\mathcal{L}(P)$ *is also* 2*-D.*
(2) $\mathcal{L}\mathcal{L}\mathcal{L}(P) = P$.

Proof. Proposition 5.3.(2) and Theorem 6.2. □

Most 2-D 3-central spaces do not contain standard pairs of generators. When P does contain such a pair, we may write $P = Fx + Fy$ where $x \rightsquigarrow y$, and then $\mathcal{L}(P) = (Fx^2 + V_1(x)) \cap (Fy^2 + V_1(y)) = \operatorname{span}\{x^2, y, xy, x^2y\} \cap \operatorname{span}\{y^2, x^2, x^2y, x^2y^2\} = \operatorname{span}\{x^2, x^2y\}$ by Proposition 5.3.(2). So from the pair $x \rightsquigarrow y$ we get the pair $x^2y \rightsquigarrow x^2$. Applying $\mathcal{L}$ again gives the pair $y^2 \rightsquigarrow xy^2$, and applying it once more brings us back to $x \rightsquigarrow y$, as the theorem predicts.

13. Linkage

Specializing Theorem 7.4 to the composition algebra $(C, \star)$ associated to A, we obtain the following chain of spaces:

Theorem 13.1. *Let* V, V' *be two maximal* 3*-central spaces in* A.

(1) *If* V *and* V' *have the same kind, then there is a maximal* 3*-central space* U *(of the same kind) such that* $V \cap U$ *and* $U \cap V'$ *are* 2*-D.*
(2) *If* V *and* V' *have opposite kinds, then there are maximal* 3*-central spaces* U *and* U' *such that* $V \cap U'$, $U' \cap U$ *and* $U \cap V'$ *are* 3*-D.*

As a final application, we deduce from Theorem 8.8 a chain for elements:

Theorem 13.2. *If* x_0 *and* x_4 *are* 3*-central elements of* A*, then there are* x_1, x_2, x_3 *such that* x_i, x_{i+1} *form a standard pair of generators for* $i = 0, 1, 2, 3$.

This was shown by Rost in [10]. In light of Proposition 12.1, this result generalizes to Theorem 8.8 and Proposition 8.5. See [5] for another treatment of chains of 3-central elements, using the Okubo product in a direct manner. An analog in characteristic 3, with somewhat longer chains, was given in [13].

14. Cyclic algebras with involution of the second kind

Fix a quadratic extension K/F, and let A be a central simple K-algebra. An involution of A is of the second kind over K/F, if the subfield of symmetric elements in K is equal to F. The existence of such an involution is equivalent to the corestriction $\operatorname{cor}_{K/F} A$ being trivial.

A classical result of Albert characterizes quaternion K-algebras with an involution of the second kind over K/F, as those defined over F, namely those of the form $K \otimes_F Q_0$ where $Q_0 = (\alpha, \beta)_{2,F}$ for $\alpha, \beta \in F$.

We begin by noting that when $K = F[\rho]$, an analogous result holds for algebras of degree 3. Clearly, an algebra of odd degree with trivial corestriction cannot be the restriction of a central simple algebra over F, as in the case of quaternions, unless itself trivial. However we prove the following:

Theorem 14.1. *Let A be a cyclic algebra of degree 3 over $K = F[\rho]$, with trivial corestriction along the extension K/F. Then, for a suitable involution of the second kind over K/F, A has a standard pair of symmetric generators. In particular, A can be presented in the form $(\alpha, \beta)_K$, where $\alpha, \beta \in F$.*

Proof. The first step is to show that the algebra has a 3-central element x with $x^3 \in F$. This is proved in [4], for any quadratic field extension K/F; for convenience we briefly sketch the proof here. The argument of [3] constructs elements x with $\mathrm{Trd}(x) = \mathrm{Trd}(x^{-1}) = 0$. By starting this construction with a subfield which is symmetric with respect to some involution of the second kind, τ, one may assume the resulting element x is a product of two τ-symmetric elements. But conjugation of the involution by a symmetric element induces again an involution (of the second kind over K/F), so x is symmetric with respect to some involution, with respect to which $x^3 \in K$ is symmetric; hence $x^3 \in F$.

By Skolem-Noether we may write $A = K[x, y]$ where $yxy^{-1} = \rho x$. Applying τ to the conjugation relation, we get $y^{-\tau}xy^{\tau} = \rho^{-1}x$, so $y^{\tau} = uy$ for some $u \in K[x]^{\times}$. Letting σ denote the automorphism of $K[x]$ induced by conjugation by y, we have that $y = y^{\tau\tau} = y^{\tau}u^{\tau} = u\sigma\tau(u)$, and since $(\sigma\tau)^2 = 1$, Hilbert's theorem 90 provides us with an element $v \in K[x]^{\times}$ for which $u = v^{-1}\sigma\tau(v)$.

Let $y' = yv^{\tau}$: conjugation by y' induces σ on $K[x]$, and $(y')^{\tau} = vy^{\tau} = vuy = \sigma(v^{\tau})y = yv^{\tau} = y'$. Thus x, y' are a standard pair of symmetric generators, and $(y'^3)^{\tau} = y'^3$ is in F. □

Remark 14.2. The argument in the first paragraph of the proof of Theorem 14.1 shows that for any involution τ of the second kind, and for any 3-central element $x \in A$, $x^3 \in F$ if and only if x is a product of two τ-symmetric elements.

15. The corresponding composition algebra

Again assume $K = F[\rho]$, and let A be a central simple algebra of degree 3 over K, with an involution of the second kind $(*)$ fixing F. The space $C_* = \mathrm{Sym}(A, *)_0$, of zero trace symmetric elements in A, is closed under the operation 10.1, and indeed $(C_*, \star)$ is a symmetric composition algebra of dimension 8 over F.

Since $(*)$ switches ρ and ρ^2 and inverts the order of the standard product, it readily follows that $(*)$ is an automorphism of $(C, \star)$. The fixed subalgebra is C_*, and in fact $K \otimes_F C_* = C$, with $\tau \otimes_F 1 = (*)$.

It follows that the restriction of the quadratic norm form from C to C_* is a 3-fold Pfister form over F. An algebra A may have two involutions $(*)$ and $(*')$ of the second kind, such that $(C_{*'}, N)$ is anisotropic while $(C_{*'}, \mathrm{N})$ is hyperbolic; this is demonstrated for $\mathrm{M}_3(\mathbb{C})$ in [6, Example 34.40].

However, by Theorem 14.1 there is always an involution $*$ for which A is generated by symmetric elements x and y such that $yx = \rho xy$. For such an involution, the form is hyperbolic, and

$$C_* = \mathrm{span}_F \left\{x, x^2, y, y^2, \rho^2 xy, \rho^2 x^2 y^2, \rho xy^2, \rho x^2 y\right\}.$$

The symmetric monomials have the form $\rho^{-ij} x^i y^j$, so it is easy to verify that $V_j(x) = F[\rho^{-j} x] y^j$. Using the multiplication table in Sect. 11.1, it follows that analogously to Remark 10.1,

$$C_* \star x = Fx^2 + F[\rho^2 x]y$$

and

$$x \star C_* = Fx^2 + F[\rho x]y^2.$$

Applying the results of the first part to $(C_*, \star)$, we obtain analogs to what was proved above, for symmetric 3-central elements in A with respect to an involution of the second kind over K/F (assuming that symmetric 3-central elements exist).

In particular we can classify maximal 3-central spaces in $\mathrm{Sym}(A, *)$, and prove Rost's Theorem 13.2 for symmetric elements:

Theorem 15.1. *Let A be a cyclic algebra of degree 3 over a field K, such that $K = F[\rho]$ is a quadratic extension of F, and let $*$ be an involution with respect to which A has symmetric 3-central elements.*

Then every maximal 3-central subspace of $\mathrm{Sym}(A, *)$ has the form $Fx + V_1(x)$ or $Fx + V_2(x)$ for some symmetric 3-central element x.

Theorem 15.2. *If x_0 and x_4 are symmetric 3-central elements of A as above, then there are symmetric x_1, x_2, x_3 such that x_i, x_{i+1} form a standard pair of generators for $i = 0, 1, 2, 3$.*

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