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Stiefel–Whitney invariants for bilinear forms ☆

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ABSTRACT

We examine potential extensions of the Stiefel–Whitney invariants from quadratic forms to bilinear forms which are not necessarily symmetric. We show that as long as the symbolic nature of the invariants is maintained, some natural extensions carry only low dimensional information. In particular, the generic invariant on upper triangular matrices is equivalent to the dimension and determinant. Along the process, we show that every non-alternating matrix is congruent to an upper triangular matrix, and prove a version of Witt's Chain Lemma for upper-triangular bases. (The classical lemma holds for orthogonal bases.)

1. Introduction

Throughout, F is a field of characteristic not 2. Let $K_*(F)$ denote the graded Milnor's K-ring of F, and put $k_*(F) = K_*(F)/2K_*(F)$. The homogeneous components are denoted by $K_n(F)$ and $k_n(F)$, respectively. As usual, $\{\alpha_1, \dots, \alpha_n\}$ denotes the class of $\alpha_1 \otimes \dots \otimes \alpha_n$ in $K_n(F)$ or in $k_n(F)$. The Stiefel–Whitney class of a non-degenerate symmetric bilinear space V, with diagonal presentation $V = \langle \alpha_1, \dots, \alpha_n \rangle$, is given by:

$$w(V) \mapsto \prod_{i=1}^{n} \bigl(1 + \{\alpha_i\}t\bigr) \in k_*(F)[\![t]\!];$$

see [2, Section I.5] for details. The value of $w(V)$ depends only on the isomorphism class of V, which we denote by $[V]$. Let $\widehat{W}(F)$ denote the Witt–Grothendieck ring consisting of formal differences of isometry classes of such forms over F. The map w naturally extends to a group homomorphism $w : \widehat{W}(F) \to 1 + tk_*(F)[\![t]\!]$ by setting $w(-[V]) = w([V])^{-1}$.

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For $V \in \widehat{W}(F)$, we decompose $w(V) = \sum_{n=0}^{\infty} w_n(V)t^n$ where $w_n : \widehat{W} \to k_n(F)$ is called the *n-th Stiefel–Whitney map*. Then

$$w_n\langle \alpha_1, \ldots, \alpha_t \rangle = \sum_{1 \leqslant i_1 < \cdots < i_n \leqslant t} \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}. \tag{1}$$

Noting that $k_1(F) \cong F^\times/(F^\times)^2$, we see that w_1 is equivalent to the determinant of a bilinear space given by

$$\det V = \det(B) \cdot \left(F^\times\right)^2 \in F^\times/\left(F^\times\right)^2, \tag{2}$$

where B is any (symmetric) matrix representing the bilinear form on V. The map w_2 is equivalent to the Hasse–Witt invariant, see [2, Rem. 5.12].

Given the significant role of the Stiefel–Whitney invariants in the theory of quadratic forms, it is natural to consider a generalization to bilinear forms in general. The purpose of this paper is to examine several potential extensions of the Stiefel–Whitney maps $\{w_n\}_{n=0}^{\infty}$, from symmetric to arbitrary regular bilinear forms. To describe our proposed extensions, let us write symbols in k_n in matrix form. If a symmetric bilinear space V is represented by a diagonal matrix $\mathrm{diag}(a_1, \ldots, a_t)$, we have that

$$w_n \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_t \end{pmatrix} = \sum_{i_1 < \cdots < i_n} \begin{Bmatrix} a_{i_1} & & \\ & \ddots & \\ & & a_{i_n} \end{Bmatrix}.$$

To generalize this, let Λ_n denote the free abelian group generated by formal symbols of the form $\{A\}$ for $A \in \mathrm{M}_n(F)$. Let $\widehat{w}_n$ be the map $\mathrm{M}(F) \to \Lambda_n$, where $\mathrm{M}(F) = \bigcup_{t=0}^{\infty} \mathrm{M}_t(F)$, defined by

$$\widehat{w}_n \begin{pmatrix} a_{11} & \ldots & a_{1t} \\ \vdots & \ddots & \vdots \\ a_{t1} & \ldots & a_{tt} \end{pmatrix} = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant t} \begin{Bmatrix} a_{i_1 i_1} & \ldots & a_{i_1 i_n} \\ \vdots & \ddots & \vdots \\ a_{i_n i_1} & \ldots & a_{i_n i_n} \end{Bmatrix}. \tag{3}$$

Now let $\mathcal{A}$ be some set of regular matrices over F, possibly of varying dimensions. We say that $A, B \in \mathcal{A}$ are congruent, and denote $A \sim B$, if they represent isomorphic bilinear forms, namely if $B = PAP^{\mathrm{t}}$ for some $P \in \mathrm{GL}(F)$.

Let $\Lambda_n^{\mathcal{A}}$ denote the subgroup of Λ_n generated by the image $\widehat{w}_n(\mathcal{A})$.

Definition 1.1. A *symbolic invariant* of dimension n for $\mathcal{A}$ is a map f from $\mathcal{A}$ to a quotient of $\Lambda_n^{\mathcal{A}}$, which factors through $\widehat{w}_n$ and is well defined up to congruence.

Thus a symbolic invariant is any map f for which there is a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\widehat{w}_n"] \arrow[d] \arrow[dr, "f"] & \Lambda_n^{\mathcal{A}} \arrow[d, dotted] \\ \mathcal{A}/{\sim} \arrow[r, dotted] & * \end{tikzcd}$$

If f is a symbolic invariant, $f(V)$ can be computed by choosing for V a representation from $\mathcal{A}$, and applying (3). For example, by (1), the Stiefel–Whitney invariant w_n is a symbolic invariant for the class $\mathcal{D}$ of diagonal matrices.

Definition 1.2. Given a set $\mathcal{A}$ of regular matrices over F, $k_n^{\mathcal{A}}$ denotes the quotient of $\Lambda_n^{\mathcal{A}}$ with respect to the relations $\widehat{w}_n(a) = \widehat{w}_n(b)$ for every $a \sim b$ in $\mathcal{A}$.

The *generic symbolic invariant* of $\mathcal{A}$ is the map $\widehat{w}_n^{\mathcal{A}} : \mathcal{A} \to k_n^{\mathcal{A}}$ induced by $\widehat{w}_n$.

Thus $\widehat{w}_n^{\mathcal{A}}$ satisfies (3); the defining relations in $k_n^{\mathcal{A}}$ are equivalent to $\widehat{w}_n^{\mathcal{A}}(a) = \widehat{w}_n^{\mathcal{A}}(b)$ whenever $a \sim b$. Therefore, any symbolic invariant factors through the generic one.

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Attempting to define symbolic invariants for all matrices, the first step would be to compute the group k_n^{GL} and construct the generic symbolic invariant for $\mathrm{GL}(F)$. It turns out that for $n = 1, 2$ this group is trivial:

Theorem 1.3. *For $n = 1$ and $n = 2$, $k_n^{\mathrm{GL}} = \mathbb{Z}$, and all symbols are equal. In fact, $\widehat{w}_n^{\mathrm{GL}}(a) = \binom{t}{n}$ for every $a \in \mathrm{GL}_t(F)$.*

Even for the class $\mathcal{S}$ of symmetric regular matrices, we still have:

Theorem 1.4. *For $n = 1$ and $n = 2$, $k_n^{\mathcal{S}} = \mathbb{Z}$.*

In addition, while the structure of the groups k_n^{GL} is not clear when $n > 2$, because of singular blocks, we can still show that as a symbolic invariant $\widehat{w}_n^{\mathrm{GL}}$ is weaker than the dimension:

Proposition 1.5. *Let $A, B \in \mathrm{GL}_t(F)$. Then, $\widehat{w}_n^{\mathrm{GL}}(A) = \widehat{w}_n^{\mathrm{GL}}(B)$ in k_n^{GL}. Moreover, all symbols in k_n^{GL} corresponding to invertible matrices are equal.*

In light of these results, and noting that the Stiefel–Whitney invariants which are defined on $\mathcal{S}$ are only symbolic with respect to $\mathcal{D}$, it makes sense to consider symbolic invariants for intermediate classes. At the same time, we want our invariant to be defined on a set having at least one representative from each congruence class.

Another issue is that we would like $k_n^{\mathcal{A}}$ to be generated by regular symbols, and for this it is necessary that principal minors of regular matrices in $\mathcal{A}$ be regular, which is not the case for the classes GL and $\mathcal{S}$.

We therefore consider the set $\mathcal{B}$ of regular upper triangular matrices. It turns out that it contains a representative of every (regular) congruence class, except at most one in every dimension, and clearly all principal minors of elements in $\mathcal{B}$ are regular. We obtain the following:

Theorem 1.6. *For every $n \geqslant 1$, $k_n^{\mathcal{B}} = \mathbb{Z} \oplus (F^\times/F^{\times 2})$, via the map $\{A\} \mapsto (1, \det A)$. In fact $\widehat{w}_n^{\mathcal{B}} : a \mapsto \left(\binom{t}{n}, \det(a)^{\binom{t-1}{n-1}}\right)$ for every $a \in \mathrm{GL}_t(F)$.*

We prove this theorem, as well as Theorems 1.3 and 1.4, in Section 2. In Section 3 we show that every non-alternating matrix is congruent to an upper triangular matrix. This is used in Section 4 to prove Proposition 1.5. Finally in Section 5 we prove a chain lemma for upper triangular spaces, which is of independent interest is spite of $k_n^{\mathcal{B}}$ being defined by low-dimensional invariants.

Notice that our definition of symbolic invariants does not require the additivity property $\widehat{w}_n(a \oplus b) = \widehat{w}_n(a) + \widehat{w}_n(b)$, although these are desirable at least for some large set of pairs a, b. In light of the previous results, $\widehat{w}_1^{\mathrm{GL}}$ and $\widehat{w}_1^{\mathcal{B}}$ are additive. Likewise if $\widehat{w}_2^{\mathrm{GL}}$ is restricted to matrices of dimension divisible by d, then it is additive modulo d^2.

Non-symbolic extensions of the Stiefel–Whitney invariant may also be considered. Let $\widehat{W}(F)$ denote the Witt–Grothendieck group consisting of formal differences (with respect to direct sum) of *symmetric* regular bilinear forms and let $\widehat{A}(F)$ denote the group obtained by taking formal differences of *arbitrary* regular bilinear forms. It was verified in [3] that the natural map $\widehat{W}(F) \to \widehat{A}(F)$ is an inclusion. A group for which the following diagram

$$\begin{tikzcd}
\widehat{A}(F) \arrow[r] & * \\
\widehat{W}(F) \arrow[u] \arrow[r, "w_n"] & k_n(F) \arrow[u]
\end{tikzcd}$$

commutes would supply an invariant generalizing the n-th Stiefel–Whitney map. The most general such invariant is the pushout of the diagram. This leads to the classification of binary forms which

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was considered by C. Riehm [6], R. Scharlau [7] and others. It is proved in [3] that the pushout is a direct sum of $k_n(F)$ with copies of some Witt–Grothendieck groups consisting of differences of hermitian forms over extensions of F. Details of this will appear elsewhere.

Finally we remark that while we work over a field, the definitions are valid over any commutative ring, and the proofs may go through for rings with sufficiently many units, as is often the case in K-theory (see for example [4]). This may serve as a tool in studying bilinear forms over rings.

2. Symbolic invariants for GL, $\mathcal{S}$ and $\mathcal{B}$

In this section we prove Theorems 1.3, 1.4 and 1.6 regarding the structure of k_n^X and $\widehat{w}_n^X$ for $X = \mathrm{GL}$, $\mathcal{S}$ and $\mathcal{B}$. We write $P * A$ for the congruence action of $\mathrm{GL}_t(F)$ on $\mathrm{M}_t(F)$, namely $P * A = PAP^{\mathrm{t}}$.

Proof of Theorem 1.3. Before starting, note that the relations of k_n^{GL} preserve the number of symbols. Therefore, it is enough to show all symbols in k_n^{GL} are equal. Recall that by definition, $\widehat{w}_n(P * A) = \widehat{w}_n(A)$ for every A and every invertible P.

Step 0. We start by showing $k_1^{\mathrm{GL}} = \mathbb{Z}$. To see this observe that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b+c+d & b+d \\ c+d & d \end{bmatrix} \tag{4}$$

for all $a, b, c, d \in F$. By applying $\widehat{w}_1$ to the middle and right matrices we get $\{a+b+c+d\} + \{d\} = \{a\} + \{d\}$ which implies $\{a+b+c+d\} = \{a\}$ for all a, b, c, d with $ad - bc \neq 0$. Take $b = c = 0$ and $a, d \neq 0$ to get $\{a+d\} = \{a\}$. This clearly implies $\{x\} = \{y\}$ for all $x, y \in F$.

Step 1. We now prove $k_2^{\mathrm{GL}} \cong \mathbb{Z}$. Apply $\widehat{w}_2$ to the middle and right matrices in (4) to get

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix} = \begin{Bmatrix} a+b+c+d & b+d \\ c+d & d \end{Bmatrix} \tag{5}$$

in k_2^{GL}, whenever $ad - bc \neq 0$. Let

$$A = \begin{bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} * A = \begin{bmatrix} a+b+a'+b' & b+b' & c+c' \\ a'+b' & b' & c' \\ a''+b'' & b'' & c'' \end{bmatrix}.$$

Then $\widehat{w}_2(A) = \widehat{w}_2(B)$. By eliminating symbols using (5) we get:

$$\begin{Bmatrix} a & c \\ a'' & c'' \end{Bmatrix} = \begin{Bmatrix} a+b+b'+a' & c+c' \\ a''+b'' & c'' \end{Bmatrix}$$

whenever $\det A \neq 0$.

Step 2. Taking $a = c = a'' = 0$ in A, we see that as long as $-a'bc'' = \det A \neq 0$,

$$\begin{Bmatrix} 0 & 0 \\ 0 & c'' \end{Bmatrix} = \begin{Bmatrix} b+b'+a' & c' \\ b'' & c'' \end{Bmatrix}.$$

As b', c', b'' are arbitrary, we can rewrite this as:

$$\begin{Bmatrix} 0 & 0 \\ 0 & c'' \end{Bmatrix} = \begin{Bmatrix} * & * \\ * & c'' \end{Bmatrix}, \quad c'' \neq 0. \tag{6}$$

In the same manner, by letting $a = c = c'' = 0$ we get:

$$\begin{Bmatrix} 0 & 0 \\ a'' & 0 \end{Bmatrix} = \begin{Bmatrix} * & c' \\ * & 0 \end{Bmatrix}, \quad c', a'' \neq 0. \tag{7}$$

Mimicking the proof of the relation (5) after reflecting A and B from step 1 across the minor diagonal, we get:

$$\begin{Bmatrix} c'' & 0 \\ 0 & 0 \end{Bmatrix} = \begin{Bmatrix} c'' & * \\ * & * \end{Bmatrix}, \quad i \neq 0. \tag{8}$$

Step 3. We now claim that for any two *non-zero* $X, Y \in \mathrm{M}_2(F)$, we have $\{X\} = \{Y\}$. Indeed, let $a, b, c, x \in F$ and assume $x \neq 0$, then by (5), (6) and (8):

$$\begin{Bmatrix} x & a \\ b & c \end{Bmatrix} = \begin{Bmatrix} x & 0 \\ 0 & 0 \end{Bmatrix} = \begin{Bmatrix} x & 0 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}, \tag{9}$$

$$\begin{Bmatrix} 0 & a \\ b & x \end{Bmatrix} = \begin{Bmatrix} 0 & 0 \\ 0 & x \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & x \end{Bmatrix} \overset{(9)}{=} \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}, \tag{10}$$

$$\begin{Bmatrix} 0 & x \\ b & 0 \end{Bmatrix} = \begin{Bmatrix} 0 & 0 \\ 1 & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 \\ 1 & 0 \end{Bmatrix} \overset{(9)}{=} \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}, \tag{11}$$

$$\begin{Bmatrix} 0 & 0 \\ x & 0 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix} \overset{(11)}{=} \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}. \tag{12}$$

Step 4. Observe that $\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right] * \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. By tensoring all matrices with the 2×2 unit matrix we obtain the congruence

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

we now apply $\widehat{w}_2$ and cancel out non-zero symbols on both sides using step 3. After doing so we are left with a zero symbol on the right and a non-zero symbol on the left, so $\left\{\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right\}$ is equal to the non-zero symbols. $\square$

Proof of Theorem 1.4. The argument of step 0 of the last proof still applies to $k_1^{\mathcal{S}}$ (assuming $b = c$) and therefore $k_1^{\mathcal{S}} \cong \mathbb{Z}$.

Since $k_2^{\mathcal{S}}$ is generated by symmetric symbols, we only need to prove all symmetric symbols are equal. Repeating the steps 1 and 2 of the proof of Theorem 1.3 with symmetric matrices, we can still obtain relations (5) and (8) (for symmetric symbols), implying that:

$$\begin{Bmatrix} a & b \\ b & c \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \tag{13}$$

whenever a or c are non-zero. Applying step 4 of that proof shows that this also holds when $a = b = c = 0$. It remains to show that

$$\begin{Bmatrix} 0 & b \\ b & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix} \tag{14}$$

for $b \neq 0$. However, it is well known that:

$$\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

when $\mathrm{char} F \neq 2$, so

$$\begin{Bmatrix} 0 & b \\ b & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & -1 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}. \qquad \square$$

Proof of Theorem 1.6. Every principal minor of a regular upper triangular matrix is regular and upper triangular, so the generators of $k_n^{\mathcal{B}}$ are upper triangular regular symbols.

Step 1. Let us first show that every symbol is equal to a diagonal one. This is clear for $n = 1$, so assume $n \geqslant 2$.

For any $m \geqslant 2$, let $P_m = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right] \oplus I_{m-2} \in \mathrm{GL}_m(F)$. Fix an arbitrary scalar $\alpha \in F^\times$, row vector $u \in \mathrm{M}_{1\times(n-1)}(F)$ and invertible matrix $A \in \mathrm{GL}_{n-1}(F)$, and consider the $(n+1)\times(n+1)$ invertible matrices

$$A' = \begin{bmatrix} \alpha & 1 & u \\ & 1 & u \\ & & A \end{bmatrix}, \qquad A'' = \begin{bmatrix} 1 & 1 & u \\ & \alpha & 0 \\ & & A \end{bmatrix}.$$

A computation shows that $P_{n+1} * A' = A''$, so $A' \sim A''$, implying that $\widehat{w}_n(A') = \widehat{w}_n(A'')$. Evaluating the equality in $k_n^{\mathcal{B}}$, there are $n+1$ symbols (of dimension n) in each side, obtained by erasing the i-th row and column for each $i = 1, \dots, n+1$. When $i > 2$ the respective minors have the same form as A' and A'', and are congruent to each other by P_n, so their symbols are equal. The symbol for $i = 2$ in the left hand side cancels out with the symbol for $i = 1$ in the right hand side, leaving the equality

$$\begin{Bmatrix} \alpha & u \\ & A \end{Bmatrix} = \begin{Bmatrix} \alpha & 0 \\ & A \end{Bmatrix}.$$

Repeating the same argument with $P_n^{(i)} = I_i \oplus \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right] \oplus I_{n-i-2}$ in place of P_n, for $i = 1, \dots, n-2$, we get:

$$\begin{aligned} \begin{Bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{n1} \\ & a_{22} & a_{23} & \dots & a_{n2} \\ & & a_{33} & \dots & a_{3n} \\ & & & \ddots & \vdots \\ & & & & a_{nn} \end{Bmatrix} &= \begin{Bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ & a_{22} & a_{23} & \dots & a_{n2} \\ & & a_{33} & \dots & a_{3n} \\ & & & \ddots & \vdots \\ & & & & a_{nn} \end{Bmatrix} \\ &= \begin{Bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ & a_{22} & 0 & \dots & 0 \\ & & a_{33} & \dots & a_{3n} \\ & & & \ddots & \vdots \\ & & & & a_{nn} \end{Bmatrix} = \cdots = \begin{Bmatrix} a_{11} & 0 & \dots & 0 \\ & a_{22} & \ddots & \vdots \\ & & \ddots & 0 \\ & & & a_{nn} \end{Bmatrix}, \end{aligned} \tag{15}$$

as asserted.

Step 2. Now let $a, b, x, y \in F^\times$. The matrices

$$C = \begin{bmatrix} a & x-a-b \\ & b \end{bmatrix}, \qquad D = \begin{bmatrix} xy^2 & y(x-a-b) \\ & abx^{-1} \end{bmatrix}$$

are congruent since $P * C = D$ where

$$P = \begin{bmatrix} y & y \\ -ax^{-1} & 1-bx^{-1} \end{bmatrix}.$$

Assume $n \geqslant 2$. For a diagonal $n \times n$ matrix E, $X \in \mathrm{M}_2(F)$ and $1 \leqslant i < j \leqslant n$, let $X_{ij,E}$ denote the $m \times m$ matrix obtained from E by spreading X in the (i,i), (i,j), (j,i) and (j,j) entries; use the same notation for symbols. The identity $P * C = D$ now gives rise to $P_{ij,I} * C_{ij,E} = D_{ij,E}$. Fixing $x = ab$ and $y = 1$ and applying $\widehat{w}_n$ to $C_{ij,E} \sim D_{ij,E}$, we get by (15) that

$$\begin{Bmatrix} a & 0 \\ & b \end{Bmatrix}_{ij,E} = \begin{Bmatrix} a & ab-a-b \\ & b \end{Bmatrix}_{ij,E} = \begin{Bmatrix} ab & ab-a-b \\ & 1 \end{Bmatrix}_{ij,E} = \begin{Bmatrix} ab & 0 \\ & 1 \end{Bmatrix}_{ij,E},$$

so fixing $i = 1$ and ranging over $j = 2, \dots, n$ we see that

$$\{A\} = \{\det A\} \tag{16}$$

where for every $c \in F^\times$, $\{c\}$ is the symbol associated to the diagonal matrix $\mathrm{diag}(c, 1, \dots, 1)$.

Repeating the argument once more with $x = a$ and $b = 1$ where a, y are arbitrary, we get

$$\begin{Bmatrix} a & 0 \\ & 1 \end{Bmatrix}_{12,I} = \begin{Bmatrix} a & -1 \\ & 1 \end{Bmatrix}_{12,I} = \begin{Bmatrix} ay^2 & -y \\ & 1 \end{Bmatrix}_{12,I} = \begin{Bmatrix} ay^2 & 0 \\ & 1 \end{Bmatrix}_{12,I}$$

so

$$\{a\} = \{ay^2\}. \tag{17}$$

Note that the relations (16) and (17) clearly hold for $n = 1$.

Step 3. To finish, fix $x = y = 1$, and notice that $C \oplus I_{n-1} \sim D \oplus I_{n-1}$ by $(P \oplus I_{n-1}) * (C \oplus I_{n-1}) = D \oplus I_{n-1}$, so $\widehat{w}_n(C \oplus I_{n-1}) = \widehat{w}_n(D \oplus I_{n-1})$. Again this is an equality of two sums of $n+1$ symbols, with the minors obtained by erasing the i-th row and column congruent in pairs for $i > 2$. After cancellation we get

$$\{a\} + \{b\} = \{ab\} + \{1\}. \tag{18}$$

It follows that for any non-zero $\alpha_1, \dots, \alpha_k$, $\{\prod_{i=1}^k \alpha_i\} + (k-1)\{1\} = \sum_{i=1}^k \{\alpha_i\}$. Let $A \in \mathcal{B}$ be a $t \times t$ regular upper triangular matrix, where $t \geqslant n$. We claim that

$$\widehat{w}_n(A) = \binom{t-1}{n-1}\{\det A\} + \binom{t-1}{n}\{1\}. \tag{19}$$

Indeed, by definition $\widehat{w}_n(A) = \sum_{N \subseteq \{1,\dots,t\}} \{A_N\}$ where A_N is the minor obtained from A by taking the rows and columns in N, and N ranges over subsets of size n. The relation (16) gives $\{A_N\} = \{\prod_{i \in N} a_{ii}\}$, so by (18) we have that $\widehat{w}_n(A) = \sum_N \{\prod_{i \in N} a_{ii}\} = \{\prod_N \prod_{i \in N} a_{ii}\} + (\binom{t}{n} - 1)\{1\} = \{(\det A)^{\binom{t-1}{n-1}}\} + (\binom{t}{n} - 1)\{1\} = \binom{t-1}{n-1}\{\det A\} + (\binom{t}{n} - \binom{t-1}{n-1})\{1\} = \binom{t-1}{n-1}\{\det A\} + \binom{t-1}{n}\{1\}$, as asserted.

To finish the proof, let us define a map $k_n^{\mathcal{B}} \to \mathbb{Z} \oplus (F^\times / F^{\times 2})$ by sending a generator $\{M\}$, for $M \in \mathcal{B}$ of dimension $n \times n$, to $(1, \det M)$. To show that this is a well defined group homomorphism, note that for every $A, B \in \mathrm{GL}_t(F)$, if $A \sim B$ then $\det(A) \equiv \det(B)$ up to squares, so by (17) and (19), both of $\widehat{w}_n(A)$ and $\widehat{w}_n(B)$ are mapped to the same element.

The map $(n, \alpha) \mapsto \{\alpha\} + (n-1)\{1\}$, which is well defined by (17), inverts the above by (16) and (18). $\square$

3. Congruence to an upper triangular matrix

In this section we show that, except when the base field has 2 elements, every non-alternating matrix is congruent to an upper triangular matrix. We do not assume $\operatorname{char} F \neq 2$ in this section. Recall that a matrix A is alternating if $A^{\mathrm{t}} = -A$ and A has zero diagonal (the latter condition is superfluous if the characteristic of F is not 2). Some of the arguments below are trivial from a geometric perspective, however we pay special attention to forms over finite fields.

Remark 3.1. A non-zero alternating matrix cannot be congruent to an upper triangular matrix. Indeed, congruence preserves alternativeness, and an upper triangular alternating matrix is necessarily zero.

It is more convenient to phrase the proofs in terms of bilinear forms and their underlying vector spaces.

Let V be a bilinear space. A vector $u \in V$ is isotropic if $(u, u) = 0$; the form is called alternating if all vectors are isotropic and totally isotropic if $(u, v) = 0$ for every u, v. A form is *upper triangular* if there is a base $b_1, \dots, b_n$ such that $(b_j, b_i) = 0$ for every $i < j$. The (ordered) base $\{b_1, \dots, b_n\}$ is then called upper triangular.

For a subset $X \subseteq V$, let $X^L = \{y\colon (y, X) = 0\}$, and similarly define $X^R = \{y\colon (X, y) = 0\}$; in particular V^L is the left radical of the form. For every $v \in V$, the dimension of $v^L = \{v\}^L$ is at least $\dim V - 1$, and equal to $\dim V - 1$ if $v \notin V^R$.

We say that a quadratic form $Q : V \to F$ is *reducible* if $Q(x) = \varphi(x)\varphi'(x)$ for some $\varphi, \varphi' \in V^*$.

Proposition 3.2. *Let $Q : V \to F$ be a quadratic form and assume there is a subspace $U \subseteq V$ of codimension* 1 *such that $Q|_U = 0$. Then Q is reducible.*

Proof. Let $v_1, \dots, v_{n-1}$ be a base of U, and let $v_n \notin U$. Then $Q(\sum \alpha_i v_i) = 0$ whenever $\alpha_n = 0$, so as a quadratic polynomial in $\alpha_1, \dots, \alpha_n$, the polynomial $Q(\sum \alpha_i v_i)$ is divisible by α_n, and the quotient is necessarily linear. $\square$

Corollary 3.3. *Let V be a non-alternating bilinear space, then there are at most two alternating subspaces of V of codimension* 1.

Proof. We may assume V has at least one alternating subspace of codimension 1. Let Q be the quadratic form corresponding to the bilinear form on V. Then, by the previous proposition, $Q(v) = \varphi_1 v \cdot \varphi_2 v$ for some $\varphi_1, \varphi_2 \in V^*$. Thus, an alternating subspace of V of codimension 1 is contained in $\ker \varphi_1 \cup \ker \varphi_2$, and must therefore be either $\ker \varphi_1$ or $\ker \varphi_2$. $\square$

Lemma 3.4. *Let V be a non-alternating bilinear space, where $F \neq \mathbb{F}_2$ and $\dim V \geqslant 3$. Then V contains at least* 3 *linearly independent anisotropic vectors.*

Proof. Let $x \in V$ be an anisotropic vector. Take $y, z \in V$ such that x, y, z are linearly independent. The polynomial $f(\alpha) = (\alpha x + y, \alpha x + y)$ is non-zero, so it has at most two roots. Since $|F| > 2$ there is an anisotropic vector $y' \in y + Fx$. By the same argument there is an anisotropic $z' \in z + Fx$, so x, y', z' are the desired vectors. $\square$

An optimal bound on the number of anisotropic vectors is given in Section 5.

Proposition 3.5. *Suppose $F \neq \mathbb{F}_2$ and let V be a bilinear space which is not totally isotropic. Then V is upper triangular if and only if V is not alternating.*

Proof. First assume V is regular. If $\dim V = 0$ or $\dim V = 1$ there is nothing to prove. In the case $\dim V = 2$ there is a vector $v \in V$ such that $(v, v) \neq 0$. Take $0 \neq u \in v^L$, then $\{v, u\}$ is an upper triangular base.

When $\dim V > 2$, there are by Lemma 3.4 linearly independent anisotropic vectors $v_1, v_2, v_3 \in V$. Since V is regular, the orthogonal spaces v_i^L are distinct, and by Proposition 3.2, at least one of them is not alternating, say v_1^L. Now, by induction on the dimension, v_1^L has an upper triangular base $\{u_2, \dots, u_n\}$, hence the ordered set $\{v_1, u_2, \dots, u_n\}$ is an upper triangular base of V.

In the general case, let $V^L = \{x\colon (x, V) = 0\}$ be the left radical of V. Let $u \in V$ be an anisotropic vector, then $u \notin V^L$, so there is a subspace $x \in U \leqslant V$ such that $V = V^L \oplus U$. Then U is regular, and an upper triangular base of U followed by a base of V^L, consists of an upper triangular base of V. $\square$

By symmetry, Proposition 3.5 is also a criterion for lower triangularity. The matrix analogue is:

Corollary 3.6. *Let $M \neq 0$ be a square matrix over a field with more than* 2 *elements, then the following conditions are equivalent*:

(i) *M is congruent to an upper triangular matrix.*
(ii) *M is congruent to a lower-triangular matrix.*
(iii) *M is not alternating.*

Example 3.7. Lemma 3.4, Proposition 3.5 and Corollary 3.6 are not true when $F = \mathbb{F}_2$ even for regular bilinear spaces. For instance, take the bilinear space obtained from the matrix:

$$\begin{bmatrix} & 1 & 1 \\ & 1 & \\ 1 & & \end{bmatrix}.$$

It is clearly regular and not alternating. However, its regularity implies that if it is upper triangular, it must have at least 3 anisotropic vectors, but only $(0,1,0)$ and $(0,1,1)$ are anisotropic. (This is the only such example in dimension 3, up to congruence.)

Remark 3.8. For symmetric bilinear spaces, Proposition 3.5 holds for an arbitrary field. This was already noted by Albert [1]. Indeed, if $F=\mathbb{F}_2$ and in the proof of Proposition 3.5 v_1^L is alternating, then v_1^L has a representation matrix of the form

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\cdots\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

As V is symmetric, this implies V has representation matrix $[\,1\,]\oplus A$. It is therefore enough to verify that

$$[\,1\,]\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}\sim[\,1\,]\oplus\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

over $\mathbb{F}_2$, via the matrix $\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\end{bmatrix}$.

4. Proof of Proposition 1.5

After proving Theorem 1.6 and its variants in Section 2, we come to prove Proposition 1.5, which is more delicate and requires Theorem 1.6, Proposition 3.5, and the following two lemmas.

Lemma 4.1. *Define the $m\times m$ matrices*:

$$A_m=\begin{bmatrix}0&1&\dots&1\\-1&\ddots&\ddots&\vdots\\\vdots&\ddots&0&1\\-1&\dots&-1&0\end{bmatrix},\qquad B_m=\begin{bmatrix}0&1&\dots&1\\-1&\ddots&\ddots&\vdots\\\vdots&\ddots&0&1\\-1&\dots&-1&1\end{bmatrix}.$$

Then $\det(B_m)=1$ for all m and $\det(A_m)$ is 0 for even m and 1 for odd m.

Lemma 4.2. *Let X be a regular alternating $m\times m$ matrix, then m is even and $X\sim A_m$ when A_m is defined as in the previous lemma.*

We can now prove Proposition 1.5.

Proof of Proposition 1.5. By Theorem 1.3 we may assume $n>2$. For $\alpha\in F$, let $\{\alpha\}$ denote the symbol $\{\operatorname{diag}(\alpha,1,\dots,1)\}$. The matrices A_m and B_m of Lemma 4.1 will be used throughout the proof.

Step 1. We claim that all non-alternating regular symbols are equal. To see this, first note that any relation in $k_n^{\mathcal{B}}$ also holds in k_n^{GL}. In particular, $\{A\}=\{\det A\}=\{\alpha^2\cdot\det A\}$ for any regular upper triangular $n\times n$ matrix A and $\alpha\in F^\times$, as noted in (16) and (17). By Proposition 3.5, every regular non-alternating matrix A is congruent to an upper triangular matrix A' and therefore

$$\{A\}=\left\{A'\right\}=\left\{\det A'\right\}=\{\det A\}\tag{20}$$

for all non-alternating $A\in\mathrm{GL}_n(F)$.

Now, let $a,b\in F^\times$ with $b\neq a$. It is easy to see that

$$X=\begin{bmatrix}a&0\\0&b-a\end{bmatrix}\quad\text{and}\quad Y=\begin{bmatrix}b&b-a\\b-a&b-a\end{bmatrix}$$

are congruent. Therefore $X \oplus I_{n-1} \sim Y \oplus I_{n-1}$, implying $\widehat{w}_n(X \oplus I_{n-1}) = \widehat{w}_n(Y \oplus I_{n-1})$. Evaluating, we have $n+1$ symbols on each side, n of them cancel immediately, leaving us with $\{b\} = \{a\}$. Thus, by (20), for all non-alternating $A, B \in \mathrm{GL}_n(F)$ we get:

$$\{A\} = \{\det A\} = \{\det B\} = \{B\}. \tag{21}$$

Step 2. We claim $\{A_n\} = \{1\}$. To this end, note that B_{n+1} has $n+1$ principal minors of size $n \times n$, n of which are equal to B_n, and one equal to A_n. By Lemma 4.1 B_n is regular so:

$$\widehat{w}_n(B_{n+1}) = \{A_n\} + n\{B_n\} \overset{(21)}{=} \{A_n\} + n\{1\}.$$

On the other hand, by Proposition 3.5 there is an upper triangular matrix U congruent to B_{n+1}. Any $n \times n$ principal minor U' of U is regular and non-alternating, hence $\{U'\} = \{1\}$. Therefore:

$$(n+1)\{1\} = \widehat{w}_n(U) = \widehat{w}_n(B_{n+1}) = \{A_n\} + n\{1\}$$

implying $\{A_n\} = \{1\}$.

Step 3. To finish, it is enough to prove that for all $A \in \mathrm{GL}_t(F)$, $\widehat{w}_n(A) = \binom{t}{n}\{1\}$. If A is not alternating, then by Proposition 3.5 there is upper triangular $A' \sim A$. The principal minors of A' are regular and non-alternating. Therefore, by step 1, $\widehat{w}_n(A) = \widehat{w}_n(A') = \binom{t}{n}\{1\}$. Finally if A is alternating, then by Lemma 4.2, $A \sim A_t$. All $n \times n$ principal minors of A_t are equal to A_n. Thus, by step 2, $\widehat{w}_n(A) = \widehat{w}_n(A_t) = \binom{t}{n}\{A_n\} = \binom{t}{n}\{1\}$. □

5. Chain lemma for upper triangular bases

In this last section we prove an upper triangular analogue of Witt's Chain Lemma about orthogonal bases:

Theorem 5.1 *(Witt). Let V be a symmetric regular bilinear space with orthogonal bases E and F, then there is a series of orthogonal bases*:

$$E = E_0, \dots, E_t = F$$

such that E_{i+1} differs from E_i by at most two adjacent vectors.

(For a proof see, for instance, [5, Lemma 58:1].)

We will use the notation of Section 3; in particular the characteristic of F is arbitrary. Recall the definition of an upper triangular base of a bilinear space V from Section 3. We call a vector $v \in V$ *upper triangular* if it is a member of some upper triangular base. Our first observation is that in spite of the inherent asymmetry of upper triangular bases, an *anisotropic* upper triangular vector can be positioned everywhere in a suitable upper triangular base.

Lemma 5.2. *Let v be an anisotropic upper triangular vector. Then for every j there is an upper triangular base $F = \{f_1, \dots, f_n\}$ such that $f_j = v$.*

Proof. We may assume $\{e_1, \dots, e_n\}$ is an upper triangular base with $e_i = v$. It is enough to prove that there are upper triangular bases $F = \{f_1, \dots, f_n\}$ and $G = \{g_1, \dots, g_n\}$ such that $f_{i-1} = e_i$ (if $i > 1$) and $g_{i+1} = e_i$ (if $i < n$). We will only check the existence of F, as the existence of G is similar. Indeed, $U = \mathrm{span}\{e_{i-1}, e_i\}$ $(i > 1)$ and let $0 \neq x \in e_i^L \cap U$. Then we can take $F = \{e_1, \dots, e_{i-2}, e_i, x, e_{i+1}, \dots, e_n\}$. (Notice $x \notin Fe_i$ since $e_i = v$ is anisotropic.) □

Proposition 5.3. *Let V be a bilinear space with* $\dim V > 1$ *and $v \in V$ an anisotropic vector. Then, v is upper triangular if and only if v^L is totally isotropic or non-alternating.*

Proof. Let $v \in V$ be anisotropic. By Lemma 5.2, v is upper triangular if and only if there exists upper triangular base $\{v, v_2, \dots, v_n\}$. Since v is anisotropic, v^L is of codimension 1, so in this case $v^L = \mathrm{span}\{v_2, \dots, v_n\}$, hence v^L is upper triangular. By Proposition 3.5 this happens if and only if v^L is totally isotropic or non-alternating. □

Remark 5.4. Upper-triangularity of a vector is equivalent to lower triangularity since reversing the order turns an upper triangular base into a lower triangular one.

Clearly, if V is regular, any upper triangular vector must be anisotropic. An anisotropic vector which is not upper triangular will be called *singular*.

Corollary 5.5. *Let V be a regular upper triangular bilinear space. If $\dim V$ is even then there are no singular vectors. If $\dim V$ is odd, then there are at most two singular directions in V.*

Proof. Assume V has a singular vector v. By Proposition 5.3, v^L is alternating and since v is anisotropic, $V = Fv \oplus v^L$. This implies v^L is regular, but regular alternating bilinear spaces are even dimensional, hence $\dim V$ must be odd. By Corollary 3.3 there can be at most two alternating codimension-1-subspaces, say U_1, U_2. Therefore, all singular vectors v are contained in $U_1^R \cup U_2^R$ (since $v^L = U_i$ for some i). But V is regular hence $\dim U_1^R = \dim U_2^R = 1$, so there are at most two singular vectors up to multiplication by a scalar. □

To prove our chain lemma for upper triangular bases, we will need the following lemma about the number of points on a quadric hypersurface. Results of this type are already known. A non-geometric proof is brought here for the sake of completeness.

Lemma 5.6. *Let Q be a non-zero quadratic form on a space V over the finite field $\mathbb{F}_q$. Then, V contains at least $(q-1)^2 q^{\dim V - 2}$ anisotropic vectors.*

The bound is easily seen to be tight by choosing Q to be a reducible form.

Proof. Let T_n stand for the minimal possible number of anisotropic vectors in V when $n = \dim V$.

Assume V contains an alternating subspace of codimension 1. Then by Proposition 3.2 $(v, v) = \varphi_1 v \cdot \varphi_2 v$ for some $\varphi_1, \varphi_2 \in V^*$. It follows that the isotropic vectors of V are $\ker\varphi_1 \cup \ker\varphi_2$, so V contains at least $q^n - 2q^{n-1} + q^{n-2} = (q-1)^2 q^{n-2}$ anisotropic vectors.

Otherwise, by the isomorphism $V \cong V^*$, V has $|\mathbb{P}V| = (q^n - 1)/(q-1)$ (non-alternating) subspaces of codimension 1. Each such subspace contains at least T_{n-1} anisotropic vectors and every anisotropic vector is contained in $(q^{n-1} - 1)/(q-1)$ subspaces of codimension 1. Therefore, V contains at least

$$\left\lceil \left(\frac{q^n - 1}{q - 1}\right)\left(\frac{q-1}{q^{n-1} - 1}\right) T_{n-1} \right\rceil \geqslant qT_{n-1}$$

anisotropic vectors. Thus, $T_n \geqslant \min\{(q-1)^2 q^{n-2}, qT_{n-1}\}$. The claim follows by induction starting from $T_1 = q - 1$. □

Theorem 5.7. *Let V be a regular bilinear space over a field F with $|F| \geqslant 7$ and let E and $\overline{E}$ be two upper triangular bases of V, then there is a series of upper triangular bases*:

$$E = E_0, \dots, E_t = \overline{E}$$

such that E_{i+1} differs from E_i by at most two adjacent vectors. In addition, when $\dim V \geqslant 2$, one may assume $t \leqslant 2 \cdot 3^{\dim V - 2} - 1$.

Proof. Let $n = \dim V$. Write $E = \{e_1, \dots, e_n\}$ and $\overline{E} = \{f_1, \dots, f_n\}$. We write $E \sim \overline{E}$ when there is a chain of upper triangular bases connecting E and $\overline{E}$, where in every two consecutive bases all but two adjacent vectors remain unchanged.

Step 0. If $n < 3$, there is nothing to prove. Assume $n \geqslant 3$. It is enough to find upper triangular bases $E' = \{e'_1, \dots, e'_n\}$ and $\overline{E}' = \{f'_1, \dots, f'_n\}$ such that $E \sim E'$, $\overline{E} \sim \overline{E}'$ and $e'_1 = f'_1$. Then, the induction hypothesis implies that $E' \sim \overline{E}'$.

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Step 1. Let P_1 be the projection from V to $U_1 = \operatorname{span}\{e_2, \dots, e_n\}$ whose kernel is Fe_1, and let P_2 be the projection from V to $U_2 = \operatorname{span}\{f_2, \dots, f_n\}$ whose kernel is Ff_1. We claim that if there is an anisotropic vector $v \in V$ such that $P_1 v$, $P_2 v$ are upper triangular in U_1, U_2 respectively, then there are E', $\overline{E}'$ as in step 0 having v as their first vector.

Indeed, if such v exists, then by Proposition 5.2, U_1 has an upper triangular base $\{P_1 v, v_1, \dots, v_{n-2}\}$, which is connected to $\{e_2, \dots, e_n\}$ by induction on n. In particular $E'' = \{e_1, P_1 v, v_1, \dots, v_{n-2}\}$ is an upper triangular base of V connected to E. Now, notice that $v \in \operatorname{span}\{e_1, P_1 v\}$ an is anisotropic. Let $0 \neq v' \in \operatorname{span}\{e_1, P_1 v\} \cap v^L$. Then $E' = \{v, v', v_1, \dots, v_{n-2}\}$ is upper triangular and clearly connected to E'', hence to E. A similar argument constructs $\overline{E}'$ with the analogous properties.

We are thus reduced to show the existence of v as above.

Step 2. If $n = 3$, U_1 and U_2 are of dimension 2 and hence (by Corollary 5.5) contain no singular vectors. This means that we only need to find an anisotropic v such that $P_1 v$ and $P_2 v$ are anisotropic as well. To this end, notice that each of the maps $x \mapsto (x, x)$, $x \mapsto (P_1 x, P_1 x)$ and $x \mapsto (P_2 x, P_2 x)$ is a non-zero quadratic form (take x to be e_1, e_2, f_2 respectively). When F is infinite, there is clearly a vector v such that the three are non-zero.

So assume F is a finite field of cardinality q. By Proposition 5.6 each of $\mathbb{P}U_1$, $\mathbb{P}U_2$ contains at most $2 = (q + 1) - (q - 1)$ isotropic vectors while $\mathbb{P}V$ contains at least $q(q - 1)$ anisotropic vectors. Hence, that number of vectors v in $\mathbb{P}V$ for which v, $P_1 v$ and $P_2 v$ are anisotropic is at least $q(q-1) - 2 \cdot 2q = q^2 - 5q$ which is larger than 0 when $q \geqslant 7$. Therefore, the existence of v is guaranteed.

Step 2′. Assume $n > 3$. We need to find an anisotropic $v \in V$ such that $P_1 v$, $P_2 v$ are upper triangular in U_1 and U_2, respectively; namely anisotropic and not singular. Recall that by Corollary 5.5, $\mathbb{P}U_1$ and $\mathbb{P}U_2$ each contains at most 2 singular vectors. In particular, v must be outside a finite number of algebraic sets in V of codimension at least 1. If F is infinite, we are done.

For F finite of cardinality q, a similar argument to step 2, will show that there are at least:

$$\begin{aligned} q^{n-2}(q-1) - 2q\left[\frac{q^{n-1}-1}{q-1} - q^{n-3}(q-1)\right] &= \frac{q^{n-2}}{q-1}\left[(q-1)^2 - 2q^2 + 2q^{3-n} + 2(q-1)^2\right] \\ &= \frac{q^{n-2}}{q-1}\left[q^2 - 6q + 3 + 2q^{3-n}\right] \end{aligned}$$

vectors in $v \in \mathbb{P}V$ for which v, $P_1 v$ and $P_2 v$ are anisotropic. The number of $v \in \mathbb{P}V$ such that $P_1 v$ or $P_2 v$ are singular cannot exceed $2 \cdot 2q = 4q$ and therefore, there are at least:

$$\frac{q^{n-2}}{q-1}\left[q^2 - 6q + 3 + 2q^{3-n} - 4q^{3-n}(q-1)\right] = \frac{q^{n-2}}{q-1}\left[q^2 - 6q + 3 - 4q^{4-n} + 6q^{3-n}\right]$$

vectors v satisfying our conditions. When $n \geqslant 4$ this number is positive for every $q \geqslant 7$.

We finish by proving the bound on t. Let us denote by T_n the maximal distance when $\dim V = n$. Clearly $T_2 = 1$. The proof uses the chains from E to E'', then to E', then to $\overline{E}'$, then to $\overline{E}''$ (similarly to E'') and from there to $\overline{E}$. The distances from E to E'', from E' to $\overline{E}'$ and from $\overline{E}''$ to $\overline{E}$ are at most T_{n-1}, and the other pairs are of distance 1. Thus, $T_n \leqslant 3T_{n-1} + 2$ and the bound $T_n \leqslant 2 \cdot 3^{n-2} - 1$ follows by induction. □

Remark 5.8. The counting argument in the proof of Theorem 5.7 works only when $|F| \geqslant 7$. In dimension 3, the claim was verified directly and it does hold over the smaller fields as well.

We conclude with a matrix-form interpretation of Theorem 5.7. Let $\mathcal{P}_n$ denote the set of n-by-n matrices obtained by positioning a 2-by-2 regular block matrix along the diagonal of the identity matrix. Form a graph $\mathcal{B}_n(F)$ whose vertices are the upper triangular matrices of size n, with A, B connected by an edge if $B = PAP^{\mathrm{t}}$ for some $P \in \mathcal{P}_n$.

Theorem 5.9. *Assume $|F| \geqslant 7$. Every two congruent upper triangular matrices belong to the same connected component of $\mathcal{B}_n(F)$, and the diameter of each connected component is at most $2 \cdot 3^{n-2} - 1$.*

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