POLYNOMIAL IDENTITIES OF $\mathrm{M}_{2,1}(G)$

UZI VISHNE

ABSTRACT. We describe the multilinear identities of the super-algebra $\mathrm{M}_{2,1}(G)$ of matrices over the Grassmann algebra, in the minimal possible degree, which is 9.

1. INTRODUCTION

Let F be a field of characteristic 0. The set of identities of an (associative) F-algebra is an ideal of the (associative) free algebra over F, closed under endomorphisms. Such an ideal is called a T-ideal. Every T-ideal is the ideal of identities for the relatively free algebra with respect to itself.

The Specht problem, asking if T-ideals over F are finitely generated (as such), was settled on the affirmative by A. Kemer. Thus, to any given algebra, there is an associated finite set of multilinear identities, from which one can deduce any identity of the algebra.

Of particular importance are the verbally prime T-ideals, which are the ideals of identities of the algebras $\mathrm{M}_n(F)$, $\mathrm{M}_n(G)$ and $\mathrm{M}_{k,\ell}(G)$ ([5],[6]). Here G is the (countably generated) Grassmann algebra

$$G = F[v_1, v_2, \dots \mid v_j v_i = -v_i v_j],$$

and $\mathrm{M}_{k,\ell}(G)$ is the super-algebra of $(k+\ell)\times(k+\ell)$ matrices over G, in which the main diagonal $k \times k$ and $\ell \times \ell$ blocks are even and the off-diagonal blocks are odd, with respect to the standard $\mathbb{Z}_2$-grading of G. The tensor product of two verbally prime algebras is PI-equivalent to a verbally prime algebra. Every PI-algebra satisfies the identities of $\mathrm{M}_n(G)$ for some n; [13, I.7.3].

In spite of their clear importance, a complete set of generators for the T-ideals of verbally prime algebras is known only in few cases: the field F, the algebra G [7], $\mathrm{M}_2(F)$ [11] and $\mathrm{M}_{1,1}(G)$ [8].

Even the PI-degree, which is the minimal degree of identities for a given algebra, is not always known. The Amitsur-Levizki theorem gives

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$\mathrm{PIdeg}(\mathrm{M}_n(F)) = 2n$, and $\mathrm{PIdeg}(G) = 3$ is fairly trivial. The degree of $\mathrm{M}_{1,1}(G)$ is 5. The degree of $\mathrm{M}_2(G)$ is 8 by [14]. The main result of the current paper is that $\mathrm{PIdeg}(\mathrm{M}_{2,1}(G)) = 9$.

An upper bound $\mathrm{PIdeg}(\mathrm{M}_n(G)) \leq (n^2+1)^2$ and $\mathrm{PIdeg}(\mathrm{M}_{k,\ell}(G)) \leq (k^2+\ell^2+1)(2k\ell+1)$ follows from the Berele-Regev theory on identities in wide hooks [1], but is clearly not tight. Popov has shown [10] that $\mathrm{PIdeg}(\mathrm{M}_{n+1}(G)) \geq \mathrm{PIdeg}(\mathrm{M}_n(G)) + 4$; he informs me that a similar method yields the lower bound $\mathrm{PIdeg}(\mathrm{M}_{p+1,q}(G)) \geq \mathrm{PIdeg}(\mathrm{M}_{p,q}(G))+3$ for $p + q \geq 2$.

Letting S_n act on multilinear identities of degree n by permuting variables, the space of identities becomes an S_n-module, isomorphic (as a module) to the group algebra $F[S_n]$. In particular the space of identities can be decomposed into irreducible components, which are modules of the respective irreducible representations of S_n.

Some identities of minimal degree of $\mathrm{M}_2(G)$ are described in [14]. For matrices over a field, the space of identities of minimal degree is one-dimensional, generated by the standard identity s_{2n}. On the other hand for $\mathrm{M}_2(G)$ the dimension is 880: there are 15 non-zero irreducible components (out of the 22 irreducible representations of S_8), with 12 of rank 1 and 3 of rank 2.

In [14, Proposition 5.3] we also reported that $\mathrm{PIdeg}(\mathrm{M}_{2,1}(G)) \geq 9$. The current paper continues [14] and describes some identities of $\mathrm{M}_{2,1}(G)$ in degree 9. As detailed below, the space of identities has dimension 4614, with 24 irreducible components (out of the 31 irreducible representations of S_9), and total rank 37. Again we see the same 'explosion' phenomenon of identities of minimal degree, which makes it hard to describe the ideal of identities in full. We describe six of the irreducible components in details.

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2. The S_n-Module of Identities

Let V_n denote the F-vector space of multilinear polynomials in the (non-commuting) variables $x_1, \dots, x_n$, and $F[S_n]$ the group algebra of the symmetric group on n letters. Permutations are multiplied as function composition, namely $(\pi\pi')i = \pi(\pi'(i))$. The map $\sigma \mapsto x_{\sigma 1} \dots x_{\sigma n}$ is a natural isomorphism of vector spaces $F[S_n] \to V_n$, and the induced left action of S_n on V_n is given by

$$\pi \cdot x_{\sigma 1} \dots x_{\sigma n} = x_{\pi\sigma 1} \dots x_{\pi\sigma n}.$$

Let $I = \mathrm{Id}_n(A)$ denote the set of multilinear identities of degree n of the algebra A. Since $\pi \cdot f(x_1, \dots, x_n) = f(x_{\pi 1}, \dots, x_{\pi n})$, this set is closed under the above left action of S_n, so that I is a submodule of V_n. We identify I with its pre-image in $F[S_n]$, which is then a left ideal.

An algorithm to compute the identities is given in [14, Section 3]. Let us sketch here the main ingredient. Identities are the elements of the group algebra in which all substitutions give the value zero. This can be interpreted, with a suitable inner product, as the orthogonal complement of the space of substitutions. In turn, substitutions are spanned by those in which every entry is a matrix unit, or a matrix unit times a generator v_i of G. See [16, Section 6.1.4] for a formal treatment of such substitutions.

Computation of the orthogonal complement is a linear algebra problem. However systems of equations with $n!$ variables (for $n = 9$) are not easy to work with, so one decomposes the problem into the irreducible representations of $F[S_n]$; the largest of these have dimension 216.

One can compute central or trace identities by essentially the same method. We report the following:

Proposition 2.1. *There are no central identities or trace identities of* $\mathrm{M}_{2,1}(G)$ *in degree* ≤ 8.

3. The multilinear identities of degree 9 of $\mathrm{M}_{2,1}(G)$

3.1. **Notation.** In order to present some of the identities, we need to set the notation. A *pattern* will be a sequence of 9 symbols from the set $\{X, Y\}$. For a pattern Π with a appearances of X and b of Y, we denote by $\Pi(x_1, \dots, x_a; y_1, \dots, y_b)$ the product of variables where the x's and the y's are in their given order, interlaced according to the pattern. For example, $XYX(x_1, x_2; y_1) = x_1 y_1 x_2$. Now let

$$C^+_\Pi(x_1, \dots, x_a; y_1, \dots, y_b) = \sum_{\sigma \in S_a, \tau \in S_b} \mathrm{sgn}(\sigma) \Pi(x_{\sigma 1}, \dots x_{\sigma a}; y_{\tau 1}, \dots, y_{\tau b}),$$

be the Capelli-type polynomial, and similarly

$$C^-_\Pi(x_1, \dots, x_a; y_1, \dots, y_b) = \sum_{\sigma \in S_a, \tau \in S_b} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \Pi(x_{\sigma 1}, \dots x_{\sigma a}; y_{\tau 1}, \dots, y_{\tau b});$$

when $b = 1$ we have $C^+_\Pi = C^-_\Pi$, and may omit the sign. On Capelli identities see for example [16, Sec. 1.2].

An identity is given as a linear combination of 9! monomials, which is would not be easy to communicate in general. Applying symmetry is therefore essential. The (projective) symmetry group $\mathrm{Sym}(f)$ of an identity f is the group of $\pi \in S_n$ such that πf is a scalar multiple of

f, necessarily $\pm f$. It is computationally easy to detect which transpositions are symmetries of a given f, and these generate a maximal subgroup of $\mathrm{Sym}(f)$ which is a direct product of symmetric groups.

Then, an action of $S_a \times S_b$ in which S_a acts via the sign operation, and S_b acts either trivially or via sign, indicates that the identity can be expressed as a combination of elements of the form C^+_Π or C^-_Π, where the weight of each pattern Π is a, b. Other actions, such as by $S_a \times S_b \times S_c$, will require patterns with more than two types of variables.

There is a standard involution on the free algebra $F\langle x_1, x_2, \dots\rangle$, defined by acting trivially on generators. The induced action on V_a is $(x_{i_1} \cdots x_{i_a})^t = x_{i_a} \cdots x_{i_1}$. Since $\mathrm{M}_{2,1}(G)$ is isomorphic to its own opposite, if f is an identity, then so is f^t. Let Π^t denote the transpose pattern, namely Π read from right to left. In its action on monomials of length a, the involution has sign $(-1)^{[\frac{a}{2}]}$, and so

$$(C^+_\Pi)^t = (-1)^{[\frac{a}{2}]} C^+_{\Pi^t}.$$

For similar reasons $(C^-_\Pi)^t = (-1)^{[\frac{a}{2}]+[\frac{b}{2}]} C^-_{\Pi^t}$, but since $a + b = 9$ is odd, we always have

$$(C^-_\Pi)^t = C^-_{\Pi^t}.$$

We define for $\epsilon \in \{+, -\}$ the symmetric and anti-symmetric elements

$$\widehat{C}^\epsilon_\Pi = C^\epsilon_\Pi + (C^\epsilon_\Pi)^t,$$

and

$$\widetilde{C}^\epsilon_\Pi = C^\epsilon_\Pi - (C^\epsilon_\Pi)^t.$$

To further save space, we encode a pattern by the indices where the Y's are. For example, $C^+_{YXXYXXXXX}$ will be written as C^+_{14}. The rest of the notation follows: since $(YXXYXXXXX)^t = XXXXXYXXY$, with an odd number of X's, we have that $(C^+_{14})^t = -C^+_{69}$, and so $\widehat{C}^+_{14} = C^+_{14} - C^+_{69}$ and $\widetilde{C}^+_{14} = C^+_{14} + C^+_{69}$.

3.2. **Explicit identities.** Let $L \leq V_9$ denote the ideal of identities of degree 9 of the algebra $\mathrm{M}_{2,1}(G)$. Table 1 below gives the rank of the component of L in every one of the 31 irreducible representations. The partition, as usual, encodes a Young diagram by giving the row lengths. Thus $(1, 1, \dots, 1) \vdash 9$ is the sign representation, and the trivial representation is $9 \vdash 9$.

We will describe the identities corresponding to the hook diagrams, as well as one other diagram. In other cases the group action leaves too many orbits, which require too many summands of the form C_Π, making it impractical to present in print. For example, the generating identities corresponding to the partition $(5, 2, 1, 1) \vdash 9$ have (up to the action $S_4 \times S_3 \times S_2$) 1210 monomials, with close to a hundred distinct integral coefficients.

The identities of $\mathrm{M}_{2,1}(G)$ in the component $(2,1,1,1,1,1,1,1) \vdash 9$ are generated by

$$T_1 = C_{XXXXXXYXX} - 2C_{XXXXYXXXX} + C_{XXYXXXXXX}.$$

Namely $Y_1 = C_7 - 2C_5 + C_3$, which we can also write as $T_1 = \widehat{C}_3 - \widehat{C}_5$.

This identity can be presented in the form of sum of commutators, as

$$T_1 = \sum_{\sigma \in S_8} \mathrm{sgn}(\sigma) x_{\sigma 1} x_{\sigma_2} [x_{\sigma 3} x_{\sigma 4}, [x_{\sigma 5} x_{\sigma 6}, y]] x_{\sigma 7} x_{\sigma 8}.$$

The identity in the component $(3,1,1,1,1,1,1) \vdash 9$ is

$$T_2 = \begin{array}{c} -\widehat{C}^+_{1,3} - 9\widehat{C}^+_{1,4} + 2\widehat{C}^+_{1,5} + 18\widehat{C}^+_{1,6} - \widehat{C}^+_{1,7} - 9\widehat{C}^+_{1,8} \\ +\widehat{C}^+_{2,3} + 7\widehat{C}^+_{2,5} - 17\widehat{C}^+_{2,7} - \widehat{C}^+_{3,4} + 3\widehat{C}^+_{3,5} - 10\widehat{C}^+_{3,6} - 2\widehat{C}^+_{4,5} \end{array}.$$

In the component $(2,2,2,1,1,1) \vdash 9$ the generating identity T_3 is a sum of elements of the form $C^-_{i,j,k}$, and $T_3^t = T_3$. So in fact T_3 is a sum of elements of the form $\widehat{C}^-_{ijk}$; note that for symmetric sets of indices we have $\widehat{C}_{i,5,10-i} = 2C^-_{i,5,10-i}$. Then

$$T_3 = \begin{array}{c} 4\widehat{C}^-_{123} + 3\widehat{C}^-_{124} - 8\widehat{C}^-_{125} - 6\widehat{C}^-_{126} + 4\widehat{C}^-_{127} + 3\widehat{C}^-_{128} - 2\widehat{C}^-_{134} \\ -5\widehat{C}^-_{135} - 5\widehat{C}^-_{136} + \widehat{C}^-_{137} + 4\widehat{C}^-_{138} + 5\widehat{C}^-_{139} - 8\widehat{C}^-_{145} + 4\widehat{C}^-_{147} \\ -9\widehat{C}^-_{148} + 4\widehat{C}^-_{156} + 4\widehat{C}^-_{157} + \widehat{C}^-_{158} - \widehat{C}^-_{234} + 8\widehat{C}^-_{235} + 11\widehat{C}^-_{236} \\ -7\widehat{C}^-_{237} - 7\widehat{C}^-_{238} - 2\widehat{C}^-_{239} - 7\widehat{C}^-_{245} - 4\widehat{C}^-_{247} + 9\widehat{C}^-_{249} - 4\widehat{C}^-_{256} \\ -4\widehat{C}^-_{257} - 4\widehat{C}^-_{345} - \widehat{C}^-_{346} + 8\widehat{C}^-_{347} + 8\widehat{C}^-_{348} + 4\widehat{C}^-_{349} - \widehat{C}^-_{356} \\ -5\widehat{C}^-_{159} + 4\widehat{C}^-_{258} + \widehat{C}^-_{456} \end{array}.$$

For $(4,1,1,1,1,1) \vdash 9$ the generating identity is a sum of elements of the form C^+_{ijk}, but is anti-symmetric. We get

$$T_4 = \begin{array}{c} \widetilde{C}^+_{124} - 2\widetilde{C}^+_{126} + \widetilde{C}^+_{128} - \widetilde{C}^+_{135} + \widetilde{C}^+_{136} - \widetilde{C}^+_{137} - 2\widetilde{C}^+_{138} \\ +\widetilde{C}^+_{139} + 2\widetilde{C}^+_{146} - \widetilde{C}^+_{148} + 2\widetilde{C}^+_{157} + \widetilde{C}^+_{158} - \widetilde{C}^+_{234} + 2\widetilde{C}^+_{235} \\ +\widetilde{C}^+_{236} - \widetilde{C}^+_{237} + \widetilde{C}^+_{238} - \widetilde{C}^+_{245} - \widetilde{C}^+_{249} \\ -2\widetilde{C}^+_{256} - \widetilde{C}^+_{346} + 2\widetilde{C}^+_{348} - \widetilde{C}^+_{356} + 2\widetilde{C}^+_{456} - 2\widetilde{C}^+_{159} \end{array}$$

The component in the partition $(5,1,1,1,1) \vdash 9$ is of rank 2, so we provide two generating identities. The group $S_5 \times S_4$ acts on both identities via the sign representation in the first component, and trivially in the second. So the identities are sums of monomials of the form $C^+_{ijk\ell}$,

partition $\vdash 9$	$\dim(\rho)$	rank $\rho(L)$	Action by sign	Trivial action
1 1 1 1 1 1 1 1 1	1	0		
2 1 1 1 1 1 1 1	8	1	S_8	
2 2 1 1 1 1 1	27	0		
2 2 2 1 1 1	48	1	$S_6 \times S_3$	
2 2 2 2 1	42	0		
3 1 1 1 1 1 1	28	1	S_7	S_2
3 2 1 1 1 1	105	2	$S_6 \times S_2$	
3 2 2 1 1	162	2	$S_5 \times S_3$	
3 2 2 2	84	1	$S_4 \times S_4$	
3 3 1 1 1	120	2	$S_5 \times S_2 \times S_2$	
3 3 2 1	168	1	$S_4 \times S_3 \times S_2$	
3 3 3	42	2	$S_3 \times S_3 \times S_3$	
4 1 1 1 1 1	56	1	S_6	S_3
4 2 1 1 1	189	2	$S_5 \times S_2$	S_2
4 2 2 1	216	3	$S_4 \times S_3$	S_2
4 3 1 1	216	3	$S_4 \times S_2 \times S_2$	
4 3 2	168	1	$S_3 \times S_3 \times S_2$	
4 4 1	84	1	$S_3 \times S_2 \times S_2 \times S_2$	
5 1 1 1 1	70	2	S_5	S_4
5 2 1 1	189	2	$S_4 \times S_2$	S_3
5 2 2	120	2	$S_3 \times S_3$	S_3
5 3 1	162	2	$S_3 \times S_2 \times S_2$	S_2
5 4	42	1	$S_2 \times S_2 \times S_2 \times S_2$	
6 1 1 1	56	1	S_4	S_5
6 2 1	105	2	$S_3 \times S_2$	S_4
6 3	48	1	$S_2 \times S_2 \times S_2$	S_3
7 1 1	28	0		
7 2	27	0		
8 1	8	0		
9	1	0		
Total rank	2620	37		
Total dimension	9!	4614		

Table 1. Decomposition of the ideal of identities of $\mathrm{M}_{2,1}(G)$ in degree 9 into irreducible representations, with a subgroup of S_9 acting on generators

and since they are antisymmetric, they can be expressed in terms of the $\widetilde{C}^+_{ijk\ell}$:

$$T_5 = \begin{array}{c} -2\widetilde{C}^+_{1235} + 4\widetilde{C}^+_{1237} - 2\widetilde{C}^+_{1239} + 4\widetilde{C}^+_{1245} - \widetilde{C}^+_{1246} - 7\widetilde{C}^+_{1247} \\ +2\widetilde{C}^+_{1248} + 4\widetilde{C}^+_{1249} - \widetilde{C}^+_{1256} - 4\widetilde{C}^+_{1257} + 5\widetilde{C}^+_{1258} + 2\widetilde{C}^+_{1259} \\ +\widetilde{C}^+_{1267} - \widetilde{C}^+_{1268} - 2\widetilde{C}^+_{1269} - 4\widetilde{C}^+_{1278} + 2\widetilde{C}^+_{1279} - 2\widetilde{C}^+_{1345} \\ -\widetilde{C}^+_{1346} + 2\widetilde{C}^+_{2346} - 2\widetilde{C}^+_{1347} + 5\widetilde{C}^+_{2347} - \widetilde{C}^+_{1348} - \widetilde{C}^+_{2348} \\ +2\widetilde{C}^+_{1349} + 2\widetilde{C}^+_{1356} - 3\widetilde{C}^+_{2356} - 3\widetilde{C}^+_{2357} + 4\widetilde{C}^+_{1357} - \widetilde{C}^+_{1358} \\ +3\widetilde{C}^+_{2358} - 3\widetilde{C}^+_{1359} - 5\widetilde{C}^+_{2367} + 9\widetilde{C}^+_{1367} + \widetilde{C}^+_{1368} - 5\widetilde{C}^+_{1369} \\ -\widetilde{C}^+_{1378} + \widetilde{C}^+_{1456} + \widetilde{C}^+_{2456} + 5\widetilde{C}^+_{2457} + 7\widetilde{C}^+_{1457} - 7\widetilde{C}^+_{3457} \\ -2\widetilde{C}^+_{2458} - 5\widetilde{C}^+_{1458} - 5\widetilde{C}^+_{1459} - 3\widetilde{C}^+_{1467} - 6\widetilde{C}^+_{2467} - 4\widetilde{C}^+_{1468} \end{array}$$

and

$$T_5' = \begin{array}{c} 99\widetilde{C}^+_{1367} + 95(\widetilde{C}^+_{1457} - \widetilde{C}^+_{1247} - \widetilde{C}^+_{3457}) + 91(\widetilde{C}^+_{1258} + \widetilde{C}^+_{2347} \\ -\widetilde{C}^+_{1369} - \widetilde{C}^+_{1458} - \widetilde{C}^+_{2367}) - 87\widetilde{C}^+_{1467} - 66\widetilde{C}^+_{2467} + 62(\widetilde{C}^+_{1245} \\ +\widetilde{C}^+_{1357} - \widetilde{C}^+_{1278} - \widetilde{C}^+_{1468}) + 58(\widetilde{C}^+_{1349} - \widetilde{C}^+_{1345} - \widetilde{C}^+_{2458}) \\ -54\widetilde{C}^+_{2368} + 37(\widetilde{C}^+_{2457} - \widetilde{C}^+_{1459}) + 33(\widetilde{C}^+_{2358} - \widetilde{C}^+_{1359} - \widetilde{C}^+_{2356} \\ -\widetilde{C}^+_{2357}) + 29(\widetilde{C}^+_{1267} + \widetilde{C}^+_{1368} + \widetilde{C}^+_{1456} + \widetilde{C}^+_{2456} - \widetilde{C}^+_{1246} \\ -\widetilde{C}^+_{1256} - \widetilde{C}^+_{1348} - \widetilde{C}^+_{1358}) + 25(\widetilde{C}^+_{1268} + \widetilde{C}^+_{1346} + \widetilde{C}^+_{1378} + \widetilde{C}^+_{2348}) \\ +8(\widetilde{C}^+_{1237} + \widetilde{C}^+_{1249} - \widetilde{C}^+_{1257}) + 4(\widetilde{C}^+_{1248} + \widetilde{C}^+_{1259} + \widetilde{C}^+_{1279} \\ +\widetilde{C}^+_{1356} + \widetilde{C}^+_{2346} - \widetilde{C}^+_{1235} - \widetilde{C}^+_{1239} - \widetilde{C}^+_{1269} - \widetilde{C}^+_{1347}) \end{array}.$$

Finally, the component corresponding to the partition $(6,1,1,1) \vdash 9$ is generated by an identity on which $S_4 \times S_5$ acts (by sign in the first component and trivially on the second), and which is antisymmetric with respect to t. It can be presented as a sum of elements of the form $\widetilde{C}^+_{ijklm}$:

$$T_6 = \begin{array}{c} -2\widetilde{C}^+_{12356} + \widetilde{C}^+_{12357} + 3\widetilde{C}^+_{12358} + \widetilde{C}^+_{12367} - 3\widetilde{C}^+_{12369} - \widetilde{C}^+_{12378} \\ -\widetilde{C}^+_{12379} + 2\widetilde{C}^+_{12389} + 2\widetilde{C}^+_{12456} - \widetilde{C}^+_{12458} + \widetilde{C}^+_{12459} - \widetilde{C}^+_{12467} \\ -2\widetilde{C}^+_{12468} + \widetilde{C}^+_{12469} - 2\widetilde{C}^+_{12478} - \widetilde{C}^+_{12479} - \widetilde{C}^+_{12489} - \widetilde{C}^+_{13457} \\ -2\widetilde{C}^+_{13458} - \widetilde{C}^+_{13459} + \widetilde{C}^+_{13467} + \widetilde{C}^+_{13468} + 3\widetilde{C}^+_{13478} + \widetilde{C}^+_{13479} \\ -\widetilde{C}^+_{23467} + \widetilde{C}^+_{23468} + 2\widetilde{C}^+_{23469} + \widetilde{C}^+_{23479} - \widetilde{C}^+_{23489} - \widetilde{C}^+_{12567} \\ -2\widetilde{C}^+_{12568} + 3\widetilde{C}^+_{12569} + 2\widetilde{C}^+_{12578} + 2\widetilde{C}^+_{12579} - \widetilde{C}^+_{13568} + \widetilde{C}^+_{13569} \\ +2\widetilde{C}^+_{13578} - 2\widetilde{C}^+_{13589} + 2\widetilde{C}^+_{14568} - \widetilde{C}^+_{14579} - 3\widetilde{C}^+_{14589} + \widetilde{C}^+_{23567} \\ -\widetilde{C}^+_{23568} - 2\widetilde{C}^+_{23579} - 2\widetilde{C}^+_{23589} - 2\widetilde{C}^+_{24569} + \widetilde{C}^+_{24578} + \widetilde{C}^+_{24579} \\ +2\widetilde{C}^+_{24589} - \widetilde{C}^+_{34578} + \widetilde{C}^+_{34589} \end{array}.$$

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Deptartment of Mathematics, Bar Ilan University, Ramat Gan 52900, Israel

E-mail address: `vishne@math.biu.ac.il`