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ARTIN COVERS OF THE BRAID GROUPS

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ABSTRACT

Computation of fundamental groups of Galois covers recently led to the construction and analysis of Coxeter covers of the symmetric groups [L. H. Rowen, M. Teicher and U. Vishne, Coxeter covers of the symmetric groups, *J. Group Theory* **8** (2005) 139–169]. In this paper we consider analog covers of Artin's braid groups, and completely describe the induced geometric extensions of the braid group.

Keywords: Artin groups; braid groups; fundamental groups.

Mathematics Subject Classification 2010: 20F36, 20F34, 14Q10

1. Introduction

The purpose of this paper is to introduce and study quotients of connected Artin groups, with their action on the space of directed paths in the unit disk. We start by explaining some of the motivation, coming from algebraic geometry.

Let X be a projective surface, with a generic projection of degree n to $\mathbb{CP}^2$. Let S denote the branch curve. The fundamental group $\pi_1(\mathbb{CP}^2 - S)$ has a natural monodromy map to Artin's braid group B_n.

Applying van Kampen's theorem, one may find a standard set of generators $\Gamma_1, \ldots, \Gamma_m$ for $\pi_1(\mathbb{CP}^2 - S)$, and the associated presentation, endowed with an epimorphism $\pi_1(\mathbb{CP}^2 - S) \to S_n$, where each Γ_j maps to a transposition. We then have a short exact sequence

$$1 \longrightarrow \pi_1(X_{\mathrm{Gal}}) \longrightarrow \pi_1(\mathbb{CP}^2 - S)/\langle\langle \Gamma_j^2 \rangle\rangle \longrightarrow S_n \longrightarrow 1,$$

where X_{Gal} is the Galois cover of X with respect to the given generic projection. Here $\langle\langle \Gamma_j^2 \rangle\rangle$ stands for the normal closure of $\langle \Gamma_j^2 \rangle$. Even with this presentation, it is still quite difficult to compute $\pi_1(\mathbb{CP}^2 - S)$, or even $\pi_1(\mathbb{CP}^2 - S)/\langle\langle \Gamma_j^2 \rangle\rangle$, e.g. to the level of deciding whether or not the latter is virtually solvable. Many special cases were computed by Moishezon, Teicher and others (see, for example, [10, 11]).

A more general approach was recently suggested in [1, 2]. Let X_0 be the degeneration of X into a union of planes, and S_0 be the union of the lines of intersection. Take T to be the dual graph, in which the vertices correspond to planes in X_0, and the edges correspond to lines in S_0. One can associate a Coxeter group $\mathrm{C}(T)$ to T, with a natural cover $\mathrm{C}(T) \to S_n$. Furthermore, a certain quotient $\mathrm{C}_{\mathrm{Y}}(T)$, still covering S_n, can be computed explicitly; this was done in [12], and we sketch the main results in Sec. 2 below. This method was successfully implemented for the case $X = \mathbb{T} \times \mathbb{T}$ (where $\mathbb{T}$ is the projective torus), where the van-Kampen presentation of $\pi_1(\mathbb{CP}^2 - S)$ has 54 generators and more than 1700 relations. Using an explicit description of $\mathrm{C}_{\mathrm{Y}}(T)$ (for an appropriate graph T), this group was shown to be virtually nilpotent of class 3 [3].

In this paper we study a group $\mathrm{A}_{\mathrm{Y}}(T)$ analogous to $\mathrm{C}_{\mathrm{Y}}(T)$, which naturally projects onto B_n (for the definition see Sec. 3). This group appears (implicitly) in a description of presentations of the braid group arising from planar graphs [13].

For certain surfaces X, one would then obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{A}_{\mathrm{Y}}(T) \arrow[r] \arrow[d] & \pi_1(\mathbb{CP}^2 - S) \arrow[r] \arrow[d] & B_n \arrow[d] \\
\mathrm{C}_{\mathrm{Y}}(T) \arrow[r] & \pi_1(\mathbb{CP}^2 - S)/\langle\langle \Gamma_j^2 \rangle\rangle \arrow[r] & S_n
\end{tikzcd}$$

In particular, the kernel of $\pi_1(\mathbb{CP}^2 - S) \to B_n$ is a quotient of the kernel $\mathrm{A}_{\mathrm{Y}}(T) \to B_n$, which we compute here in detail. Let us also mention that the defining relations of $\mathrm{A}_{\mathrm{Y}}(T)$ appear in a similar context in [8]. A description of the fundamental group of the discriminant complement of a versal unfolding of a Brieskorn–Pham polynomial $x_1^{l_1+1} + \cdots + x_n^{l_n+1}$ was given in [9], and one finds there too the same defining relations. Our goal in this paper is mostly group theoretic, so we do not pursue applications to algebraic geometry any further.

Another quotient of the standard braid group, with respect to the normal subgroup generated by the commutator $[\sigma_1, \sigma_2^2\sigma_3\sigma_2^{-2}]$, was computed by Teicher (see [14]), and shown to be an extension of the symmetric group by a solvable group. We thank Prof. Teicher for useful conversations on this and other topics.

Note. Throughout the paper, composition of functions is performed in the usual order, namely $(f \circ g)(x) = f(g(x))$; however the action of S_n or the braid group B_n is reversed: $(\sigma\tau)(u) = \tau(\sigma(u))$.

The paper is organized as follows. In Sec. 2 we review the construction of Coxeter covers of the symmetric group and the main results from [12]: to a connected undirected graph T one associates a Coxeter group $\mathrm{C}(T)$ whose quotient $\mathrm{C_Y}(T)$ is a semidirect product of S_n and a subgroup of $\mathbb{F}_n^m$ for m the rank of $\pi_1(T)$. Section 3 is devoted to the definition and basic properties of the groups $\mathrm{A}(T)$ and $\mathrm{A_Y}(T)$, where T is an arbitrary planar graph (given with an embedding in the plane). The defining relations of $\mathrm{A_Y}(T)$ are defined from the local neighborhoods of the graph. One interesting feature of this particular construction is that $\mathrm{A_Y}(T')$ is a retract subgroup of $\mathrm{A_Y}(T)$ for any connected spanning subgraph $T' \subseteq T$ (Theorem 3.11).

In Sec. 4 we recall the action of the braid group on the disk and set the basic notations.

In Sec. 5 we define geometric extensions of a group acting transitively on a space. Using this special type of HNN extensions, we construct maximal quotients of groups with respect to certain geometric data. Our main interest is in the maximal quotient $G(T)$ of $\mathrm{A_Y}(T)$ which is a geometric extension of B_n with respect to its action on directed paths in the disk. These are precisely groups arising in the above mentioned algebraic-geometry context.

Then, in Sec. 6 we compute $G(T^{(1)})$ explicitly for the cycle graph $T^{(1)}$. After discussing geometric extensions on quotient spaces in Sec. 7, we compute some quotient groups of related actions. For example, it turns out that for the action of $A(T^{(1)})$ forgetting the direction of paths is equivalent to forgetting the whole path except for its endpoints.

In order to apply the computation of $G(T^{(1)})$ to the general case, we show in Sec. 8 that $\mathrm{A_Y}(T)$ depends on T only through combinatorial data, thus allowing one to choose the graph structure at will. Likewise we show in Sec. 9 that the same property holds for $G(T)$.

Finally, in Sec. 10, we compute $G(T)$ for an arbitrary planar graph: $G(T) = B_n \ltimes K_{n,m}$, where n is the number of vertices in T and m is the rank of $\pi_1(T)$. The kernel $K_{n,m}$ is a central extension of a certain canonical subgroup of $\mathbb{F}_n^m$, by the elementary Abelian group $(\mathbb{Z}/2\mathbb{Z})^m$. In particular the word problem in $G(T)$ is solvable, enabling practical computation of quotients that arise in practice.

2. Coxeter Covers of the Symmetric Groups

This paper generalizes [12] from Coxeter covers of the symmetric groups to Artin covers of the braid group. Therefore, let us quickly review some definitions and main results of that paper.

The standard way to associate a Coxeter group to a (simple) Dynkin diagram is to associate a generator to each vertex, and impose the relations $uv = vu$ when two vertices are connected by an edge. Our definition is a dual one.

Definition 2.1. Let T be an undirected, simple graph on n vertices, with no loops. We define the Coxeter group $\mathrm{C}(T)$ as the abstract group whose formal generators are the edges of T, with the relations $u^2 = 1$ for every edge $u \in T$, $uv = vu$ if u, v are disjoint (i.e. no common vertex), and $uvu = vuv$ if u, v share a common vertex.

Not every Dynkin diagram can be realized in this manner. For example, if a generator x of $\mathrm{C}(T)$ does not commute with generators y_1, y_2, y_3, then the y_i cannot all commute with each other. In particular Coxeter groups of type D_n (whose corresponding Artin groups are analyzed in [6]) cannot be realized.

There is a natural epimorphism of $\mathrm{C}(T)$ to the symmetric group S_n, defined by sending an edge whose endpoints are i and j to the transposition $(i\,j)$. This map is onto if and only if T is connected.

Next, one may define the quotient $\mathrm{C}_{\mathrm{Y}}(T)$ of $\mathrm{C}(T)$, by adding the relation $[u, vwv] = 1$ whenever u, v, w form the subgraph shown in Fig. 1 below. The map $\mathrm{C}(T) \to S_n$ splits through $\mathrm{C}_{\mathrm{Y}}(T)$. The main purpose of [12] is to compute the kernel of $\mathrm{C}_{\mathrm{Y}}(T) \to S_n$.

Let m denote the number of basic cycles in T (namely m is the rank of $\pi_1(T)$). Let $\mathbb{F}_m$ denote the free group on m letters, so $\mathbb{F}_m^n$ is a direct product of n copies of this group. Let $F_{m,n}$ denote the kernel of the cumulative abelianization map $\mathrm{ab}\colon \mathbb{F}_m^n \to \mathbb{Z}^m$, defined by $\mathrm{ab}(w_1, \ldots, w_n) = \sum \mathrm{ab}_i(w_i)$, where ab_i is the usual abelianization map from the ith component in $\mathbb{F}_m^n$ to $\mathbb{Z}^m$. Thus we have a short exact sequence

$$1 \longrightarrow F_{m,n} \longrightarrow \mathbb{F}_m^n \longrightarrow \mathbb{Z}^m \longrightarrow 1. \tag{2.1}$$

Furthermore, let $A_{m,n}$ be the group generated by $x_{rs}^{(i)}$ ($i = 1, \ldots, n$, $r, s = 1, \ldots, m$) with the defining relations

$$x_{rr}^{(i)} = 1, \tag{2.2}$$

$$x_{rs}^{(i)} x_{st}^{(i)} = x_{rt}^{(i)}, \tag{2.3}$$

$$x_{st}^{(i)} x_{rs}^{(i)} = x_{rt}^{(i)}, \tag{2.4}$$

and

$$[x_{rs}^{(i)}, x_{tu}^{(j)}] = 1 \quad \text{if } r, s, t, u \text{ are distinct.} \tag{2.5}$$

It is shown in [12] that for $n \geq 5$ we have a short exact sequence

$$1 \longrightarrow F_{m,n} \longrightarrow \mathrm{C}_{\mathrm{Y}}(T) \longrightarrow S_n \longrightarrow 1, \tag{2.6}$$

and in fact that $\mathrm{C}_{\mathrm{Y}}(T) \cong S_n \ltimes F_{m,n}$, where S_n acts by permuting entries. Also, it is shown that (again when $n \geq 5$), $A_{m,n} \cong F_{m,n}$, where the isomorphism is given by $x_{rs}^{(i)} = {x_s^{(i)}}^{-1} x_r^{(i)}$. The advantage of having this isomorphism is that the word problem is obviously decidable in $F_{m,n}$ (and so it is easy to define maps into this group), while the explicit presentation of $A_{m,n}$ allows to define maps from it. In Sec. 10 we will meet the analogs of these two groups.

Proposition 2.2. *Assume $n \geq 5$. Then $F_{m,n}$ is the pullback of the diagram*

$$\begin{tikzcd} ? \arrow[r, dotted] \arrow[d, dotted] & \mathbb{F}_m \arrow[d] \\ \mathbb{F}_m^{n-1} \arrow[r] & \mathbb{Z}^m \end{tikzcd}$$

where the map $\mathbb{F}_m^{n-1} \to \mathbb{Z}^m$ is the cumulative abelianization as above, and the map $\mathbb{F}_m \to \mathbb{Z}^m$ is minus the abelianization.

Proof. It is well known that the solution to such a pullback diagram is the subgroup $\{(w,t) \in \mathbb{F}_m^{n-1} \times \mathbb{F}_m : \sum_{i=1}^{n-1} \mathrm{ab}_i(w_i) = -\mathrm{ab}(t)\}$, and this is clearly $F_{m,n}$. □

One can thus easily construct an epimorphism $F_{m,n} \to \mathbb{F}_m^{n-1}$.

3. The Local Quotient of Artin Groups

3.1. *The definition*

Let T be a planar graph on n points, by which we mean the graph is a properly labeled union of paths which do not intersect except possibly at the endpoints. The graph is not necessarily simple (namely two edges may share the same two endpoints), but we assume throughout that T has no loops, namely every edge connects two distinct vertices. By an *isomorphism* of graphs we mean a deformation of (a compact domain in) the plane which carries one graph to the other.

We view T as the set of its edges. Throughout the paper, we denote $[u,v] = (uv)(vu)^{-1}$ and $\langle u,v\rangle = (uvu)(vuv)^{-1}$.

Definition 3.1. Let us define a group $\mathrm{A}(T)$ with the edges of T as generators, and the relations

$$[u,v] = 1 \quad \text{if } u,v \text{ are disjoint in } T, \tag{3.1}$$

$$\langle u,v\rangle = 1 \quad \text{if } u,v \text{ intersect in only one vertex.} \tag{3.2}$$

In the third possible case, namely if u and v share two vertices, then no relation is assumed to hold between them.

Evidently, $\mathrm{A}(T)$ is an Artin group [4], with exponents 2 and 3. It is connected; almost always of large type; but never triangle-free, see [5]. The best known example of such a group is when T is a single path connecting n vertices; then there are

$n-1$ generators with the usual braid relations, and in this case $\mathrm{A}(T) \cong B_n$ is the standard braid group. In Theorem 3.11 (following Remark 3.4) we show that when T is connected, there is a surjection $\mathrm{A}(T) \to B_n$. This is further studied in Sec. 4, where we show that the map is given by sending a generator to the half-twist induced by the corresponding path.

In [13] the author gives a presentation of the braid group B_n on (the edges of) T, assuming T is simple. The presentation involves two families of relations apart from those defining $\mathrm{A}(T)$: one relation for every triple of edges with a common vertex, and one relation for every cycle in the graph.

Motivated by examples from algebraic geometry (related to the computation of the fundamental group of Galois covers, e.g. [1, 3]), we are interested in this paper in "local" relations, with bounded support (bounded in terms of the graph distance); thus we only assume the first family of relations. However since in general T is not simple, we also add relations for pairs of edges intersecting in two vertices.

Definition 3.2. Let T be a planar graph. The group $\mathrm{A}_{\mathrm{Y}}(T)$ is the quotient of $\mathrm{A}(T)$ obtained by adding the following relations:

$$[w^{-1}uw, v] = 1 \quad \text{if } u, v, w \text{ are as in Fig. 1,} \tag{3.3}$$

$$\langle w^{-1}uw, v\rangle = 1 \quad \text{if } u, v, w \text{ are as in Fig. 2,} \tag{3.4}$$

$$[w^{-1}uw, v^{-1}xv] = 1 \quad \text{if } x, u, v, w \text{ are as in Fig. 3,} \tag{3.5}$$

$$\langle w^{-1}uw, v^{-1}xv\rangle = 1 \quad \text{if } x, u, v, w \text{ are as in Fig. 4.} \tag{3.6}$$

Remark 3.3. The defining relations of $\mathrm{A}(T)$ are obtained by ranging over all the embeddings of the subgraphs of Figs. 1–4 in T.

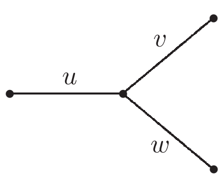

Fig. 1.

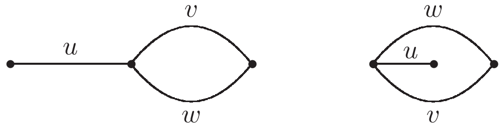

Fig. 2.

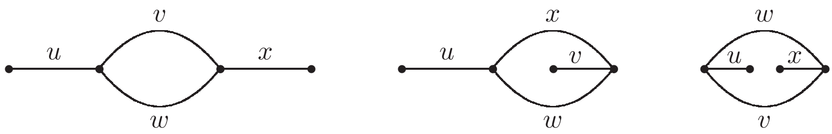

Fig. 3.

Fig. 4.

However, it suffices to take one labeling of each subgraph. More precisely, let S be a subgraph of the above-mentioned forms, and let τ be a graph automorphism of S (reflections are allowed). Then the relation induced by $\tau(S)$ is conjugate to the relation induced by S in the group $\mathrm{A}(T)$.

Proof. If $u, v, w \in T$ and u, v have a common vertex, then

$$\begin{aligned}[u^{-1}wu, v] &= u^{-1}[w, uvu^{-1}]u \\ &= u^{-1}[v^{-1}uv, w]^{-1}u,\end{aligned}$$

so u, v, w of Fig. 1 can be cyclically permuted. If u, v, w are as in the left-hand side of Fig. 2, then $\langle uwu^{-1}, v\rangle = 1$ implies $\langle u^{-1}vu, w\rangle = 1$ by conjugating with u. If x, u, v, w are as in Fig. 3, then $xu = ux$ and so

$$[x^{-1}wx, u^{-1}vu] = x^{-1}u^{-1}[uwu^{-1}, xvx^{-1}]ux.$$

The other cases are similar. □

The relations added here do not interfere with the interpretation of the elements of $A(T)$ as braids. In other words, the map $\mathrm{A}(T) \to B_n$ mentioned above induces a well-defined map $\mathrm{A}_{\mathrm{Y}}(T) \to B_n$. Moreover from [13] it follows that, assuming T is simple, in order to obtain a presentation of the standard braid group B_n, one has to add a single relation for every cycle in T; in other words, the kernel of $\mathrm{A}_{\mathrm{Y}}(T) \to B_n$ is the normal subgroup generated by certain *cyclic* words.

If T has no cycles then we have the following from [13]; the claim also follows easily from Theorem 8.3.

Remark 3.4. If T is a (planar) tree, then $\mathrm{A}_{\mathrm{Y}}(T) \cong B_n$.

Notice that if T is a simple graph, only the relations of type (3.3) appear. We also remark that adding the relations $u^2 = 1$ for every $u \in T$, turns $\mathrm{A}(T)$ into a

$$\begin{tikzcd}
\mathrm{A}(T) \arrow[r] \arrow[d] & \mathrm{A}_{\mathrm{Y}}(T) \arrow[r] \arrow[d] & B_n \arrow[d] \\
\mathrm{C}(T) \arrow[r] & \mathrm{C}_{\mathrm{Y}}(T) \arrow[r] & S_n
\end{tikzcd}$$

Fig. 5.

Coxeter group. Moreover $\mathrm{A}_{\mathrm{Y}}(T)$ projects to the group $\mathrm{C}_{\mathrm{Y}}(T)$ described in Sec. 2 (for T simple), and we have the commutative diagram of Fig. 5.

3.2. *Parabolicity*

Our next goal is to provide a structural explanation for the defining relations of $\mathrm{A}_{\mathrm{Y}}(T)$. We first define a useful partial action of the set of (non-oriented) paths in the plane on itself, to be elaborated upon in Sec. 4. Since we consider paths which are not contained in T, let us clarify what we mean by a path here. In the presence of a planar graph T, a path is defined up to relative isotopy within the complement of the union of the edges of the graph in the plane. When T is understood from the context, we write $\sim$ for this homotopy relation. Let us record few identities in this spirit.

Definition 3.5. Let x and y be (non self-intersecting) paths in the plane. Suppose that either x and y do not intersect, or they intersect at a single endpoint, p. In the first case we set $x \cdot y = y$. In the second case, we define $x \cdot y$ as the path obtained by traveling along x, circling p clockwise and then traveling along y (see Fig. 6).

Remark 3.6. Let T be a planar graph. We define a binary relation on the edges of T, as follows: $x \smile y$ when x and y intersect at one endpoint, p, and y follows x consecutively in the clockwise order around p.

(1) Suppose $x \smile y$. Then we have the reflexivity relations

$$y \cdot (x \cdot y) \sim (y \cdot x) \cdot y \sim x$$

and

$$(x \cdot y) \cdot x \sim x \cdot (y \cdot x) \sim y.$$

Fig. 6.

(2) Suppose $x \smile y$ and $y \smile z$, where x, z are disjoint. Then we have the associativity relation

$$x \cdot (y \cdot z) \sim (x \cdot y) \cdot z.$$

The set of relations defining $\mathrm{A_Y}(T)$ is best explained by the following construction, and the theorem that follows.

Definition 3.7. Let T be a planar graph. The graph $\hat{T}$ is defined on the same vertices. The edges of $\hat{T}$ are either *actual* or *virtual*. The actual edges are edges of T. For every ordered pair of edges $x, y \in T$ intersecting at a single common vertex, we have the virtual edge $x \cdot y$.

By construction, for every edge x and a vertex p on x, the virtual edges $x \cdot y$ ($p \in y \in T$) do not intersect in a small neighborhood of p, see Fig. 7. Likewise for the edges $y \cdot x$ ($p \in y \in T$).

Although we assume throughout that T is planar, one can define $\mathrm{A}(T)$ and $\mathrm{A_Y}(T)$ for any graph immersed in the plane (in general position), where the relations are only between edges which do not intersect outside the set of vertices. The group $\mathrm{A_Y}(T)$ can now be understood as the maximal quotient of $\mathrm{A}(T)$ to which the natural map from $A(\hat{T})$ is well-defined.

Theorem 3.8. *Let T be a graph embedded in the plane. There is a well-defined map $A(\hat{T}) \to \mathrm{A_Y}(T)$ sending a real edge $x \in \hat{T}$ to $x \in T$, and a virtual edge $x \cdot y$ to $x^{-1}yx$.*

Proof. There are three types of relations defining $A(\hat{T})$: relations among real edges, relations of the form $[x \cdot y, z]$ or $\langle x \cdot y, z \rangle$, and relations of the form $[x \cdot y, z \cdot u]$ or $\langle x \cdot y, z \cdot u \rangle$.

The relations from the first family are satisfied already in $\mathrm{A}(T)$. If $x \cdot y$ and z do not intersect, then either z is disjoint from x and y (and then $[x^{-1}yx, z] = 1$ in $\mathrm{A}(T)$), or x, y, z form the graph of Fig. 1, in which case $[x^{-1}yx, z] = 1$ in $\mathrm{A_Y}(T)$ by relation (3.3). Similarly if $x \cdot y$ and z share a common vertex, then either x, y, z

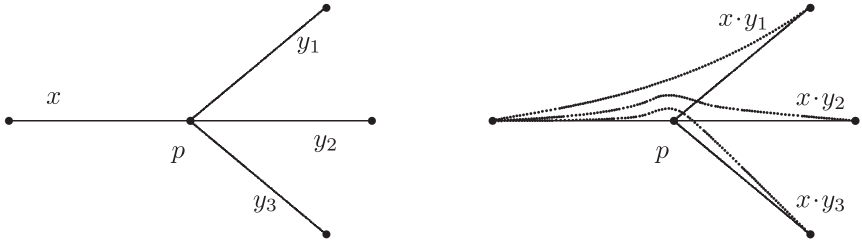

Fig. 7. Construction of $\hat{T}$.

form a path or they are as in Fig. 2. In either case it is easy to see that $\langle x^{-1}yx, z\rangle$ in $\mathrm{A_Y}(T)$. The cases $x = z$ and $y = z$ are easy.

Finally consider two virtual edges $x \cdot y$ and $z \cdot u$. If they do not intersect, the proof is either trivial or relies on relation (3.5). When they do share a common vertex, one uses relation (3.6); the only case that requires some care is to show that $\langle x \cdot y, z \cdot u\rangle = 1$ when x, z, u form a triangle and y connects a point inside the triangle to the common vertex of x and u. Then we are done by relation (3.3) applied to x, y and u. The cases where $\{x, y\} \cap \{z, u\} \neq \emptyset$ are all easy. □

Let D be a bounded connected component in the complement of T in the plane. Let $v_0, v_1, \ldots, v_n$ denote the edges on the boundary of D, traveling counterclockwise. Notice that we may have $v_i = v_j$ if both "sides" of the same edge are contained in D, as in Fig. 8. Every list of the form $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ with $j \geq i$ will be called a *planar path*. When $j > i$, the path runs from the vertex on v_i which is not on v_{i+1} to the vertex on v_j which is not on v_{j-1}. Notice that a closed proper subpath of $v_0, \ldots, v_n$ must circle a domain clockwise. For example, in Fig. 8, $v_4v_5v_6v_7v_8$ is a subpath; but $v_0v_1v_2v_{10}$ which circles a domain counterclockwise is not a subpath, as the edges are not consecutively following the boundary of D. We define

$$\mathcal{L}(v_i \cdots v_j) = v_i^{-1} \cdots v_{j-1}^{-1} v_j v_{j-1} \cdots v_i, \tag{3.7}$$

viewed as an element of $\mathrm{A}(T)$ or $\mathrm{A_Y}(T)$.

Lemma 3.9. *Let $v_1, \ldots, v_n$ be a planar path in T, running from a vertex β to a vertex γ. Assume $\beta \neq \gamma$.*

Let $x = \mathcal{L}(v_1 \cdots v_n)$. If the vertices of some $y \in T$ are disjoint from β and γ, then y commutes with x in $\mathrm{A_Y}(T)$.

Proof. Let $\alpha \neq \beta, \gamma$ be a vertex on y. We first assume the other vertex of y does not touch any of the v_i. Since the path is enumerated consecutively, the edges touching α come in pairs, v_{i_1}, v_{i_1+1}, v_{i_2}, v_{i_2+1} up to v_{i_u}, v_{i_u+1} where $i_u + 1 \leq n$ (and possibly $v_{i_j+1} = v_{i_{j+1}}$ for certain values of j). The case $u = 0$ is trivial, so assume $u \geq 1$. Note that y commutes with the v_k with $k < i_1$, and by relation (3.3), it commutes

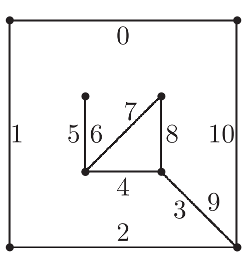

Fig. 8. A connected component with boundary.

with $v_{i_1}^{-1}v_{i_1+1}v_{i_1}$. Also, v_{i_1} commutes with $x' = \mathcal{L}(v_{i_1+2}\cdots v_n)$, by induction on u. Therefore y commutes with

$$\begin{aligned} x &= v_1^{-1}\cdots v_{i_1-1}^{-1}(v_{i_1}^{-1}v_{i_1+1}^{-1}v_{i_1})v_{i_1}^{-1}(v_{i_1+2}^{-1}\cdots v_{n-1}^{-1}v_nv_{n-1}\cdots v_{i_1+2})v_{i_1} \\ &\quad\cdot(v_{i_1}^{-1}v_{i_1+1}v_{i_1})v_{i_1-1}\cdots v_1 \\ &= v_1^{-1}\cdots v_{i_1-1}^{-1}(v_{i_1}^{-1}v_{i_1+1}^{-1}v_{i_1})v_{i_1}^{-1}x'v_{i_1}\cdot(v_{i_1}^{-1}v_{i_1+1}v_{i_1})v_{i_1-1}\cdots v_1 \\ &= v_1^{-1}\cdots v_{i_1-1}^{-1}(v_{i_1}^{-1}v_{i_1+1}^{-1}v_{i_1})x'(v_{i_1}^{-1}v_{i_1+1}v_{i_1})v_{i_1-1}\cdots v_1. \end{aligned}$$

Next, suppose the other vertex of y also touches vertices on the path x. If no v_i touches the two ends of y, the proof is basically the same. Otherwise a very similar argument can be used, with Eq. (3.5) replacing (3.3) — unless $y = v_i$, which is also an easy computation. □

Note the identities

$$\langle a, bcb^{-1}\rangle = b\langle b^{-1}ab, c\rangle b^{-1}, \tag{3.8}$$

$$\langle bab^{-1}, c\rangle = b\langle a, b^{-1}cb\rangle b^{-1}. \tag{3.9}$$

Lemma 3.10. *Let $v_1,\ldots,v_n$ be a planar path in T, running from β to γ. Suppose $y \in T$ has vertices α,β where $\alpha \neq \beta,\gamma$. Then $\langle y, \mathcal{L}(v_1\ldots v_n)\rangle = 1$. (Note that we do not assume $\beta \neq \gamma$.)*

Proof. If $n = 1$ the claim repeats relation (3.2). First assume $v_1,\ldots,v_n$ is an open loop. Then by induction $\langle v_1, \mathcal{L}(v_2\cdots v_n)\rangle = 1$. Notice also that y commutes with $\mathcal{L}(v_2\cdots v_n)$ by the previous lemma. Now, applying (3.8) and (3.9), we have

$$\begin{aligned} \langle y, \mathcal{L}(v_1\cdots v_n)\rangle &= \langle y, v_1\mathcal{L}(v_2\cdots v_n)v_1^{-1}\rangle \\ &= v_1\langle v_1^{-1}yv_1, \mathcal{L}(v_2\cdots v_n)\rangle v_1^{-1} \\ &= v_1\langle yv_1y^{-1}, \mathcal{L}(v_2\cdots v_n)\rangle v_1^{-1} \\ &= v_1y\langle v_1, y^{-1}\mathcal{L}(v_2\cdots v_n)y\rangle y^{-1}v_1^{-1} \\ &= v_1y\langle v_1, \mathcal{L}(v_2\cdots v_n)\rangle y^{-1}v_1^{-1} = 1. \end{aligned}$$

So suppose $v_1,\ldots,v_n$ is a closed path. Let β denote the intersection point of y, v_1 and v_n. Let δ denote the other point on v_n (note that this point may lay on v_1). Obviously δ is a point on v_{n-1}. First assume that none of $v_2,\ldots,v_{n-2}$ contain δ. Then v_n commutes with $v_2,\ldots,v_{n-2}$ and

$$\begin{aligned} \mathcal{L}(v_2\cdots v_n) &= v_2^{-1}\cdots v_{n-2}^{-1}v_{n-1}^{-1}v_nv_{n-1}v_{n-2}\cdots v_2 \\ &= v_2^{-1}\cdots v_{n-2}^{-1}v_nv_{n-1}v_n^{-1}v_{n-2}\cdots v_2 \\ &= v_nv_2^{-1}\cdots v_{n-2}^{-1}v_{n-1}v_{n-2}\cdots v_2v_n^{-1} \\ &= v_n\mathcal{L}(v_2\cdots v_{n-1})v_n^{-1}. \end{aligned}$$

In this case v_2 does not contain δ, and so y, v_1, v_n satisfy the condition of Eq. (3.3), and we have

$$\begin{aligned}\langle y, \mathcal{L}(v_1 \cdots v_n)\rangle &= \langle y, v_1^{-1}\mathcal{L}(v_2 \cdots v_n)v_1\rangle \\ &= \langle y, v_1^{-1}v_n\mathcal{L}(v_2 \cdots v_{n-1})v_n^{-1}v_1\rangle \\ &= v_1^{-1}v_n\langle v_n^{-1}v_1yv_1^{-1}v_n, \mathcal{L}(v_2 \cdots v_{n-1})\rangle v_n^{-1}v_1 \\ &= v_1^{-1}v_n\langle v_1yv_1^{-1}, \mathcal{L}(v_2 \cdots v_{n-1})\rangle v_n^{-1}v_1 \\ &= v_1^{-1}v_nv_1\langle y, v_1^{-1}\mathcal{L}(v_2 \cdots v_{n-1})v_1\rangle v_1^{-1}v_n^{-1}v_1 \\ &= v_1^{-1}v_nv_1\langle y, \mathcal{L}(v_1 \cdots v_{n-1})\rangle v_1^{-1}v_n^{-1}v_1\end{aligned}$$

which is trivial by induction.

Finally, let $j < n-1$ be the maximal index for which v_j contains δ. Then

$$\begin{aligned}\mathcal{L}(v_2 \cdots v_n) &= v_2^{-1} \cdots v_{n-1}^{-1}v_nv_{n-1} \cdots v_2 \\ &= v_2^{-1} \cdots v_{n-2}^{-1}v_nv_{n-1}v_n^{-1}v_{n-2} \cdots v_2 \\ &= v_2^{-1} \cdots v_j^{-1}v_n\mathcal{L}(v_{j+1} \cdots v_{n-1})v_n^{-1}v_j \cdots v_2 \\ &= v_2^{-1} \cdots v_{j-1}(v_j^{-1}v_nv_j)\mathcal{L}(v_j \cdots v_{n-1})(v_j^{-1}v_nv_j)^{-1}v_{j-1} \cdots v_2 \\ &= (v_j^{-1}v_nv_j)\mathcal{L}(v_2 \cdots v_{n-1})(v_j^{-1}v_nv_j)^{-1}.\end{aligned}$$

As before,

$$\begin{aligned}\langle y, \mathcal{L}(v_1 \cdots v_n)\rangle &= \langle y, v_1^{-1}\mathcal{L}(v_2 \cdots v_n)v_1\rangle \\ &= \langle y, v_1^{-1}(v_j^{-1}v_nv_j) \cdot \mathcal{L}(v_2 \cdots v_{n-1})(v_j^{-1}v_nv_j)^{-1}v_1\rangle \\ &= v_1^{-1}v_j^{-1}v_nv_j \\ &\quad \cdot \langle v_j^{-1}v_n^{-1}v_jv_1yv_1^{-1}v_j^{-1}v_nv_j, \mathcal{L}(v_2 \cdots v_{n-1})\rangle v_j^{-1}v_n^{-1}v_jv_1 \\ &= v_1^{-1}v_j^{-1}v_nv_j\langle v_1yv_1^{-1}, \mathcal{L}(v_2 \cdots v_{n-1})\rangle v_j^{-1}v_n^{-1}v_jv_1 \\ &= v_1^{-1}v_j^{-1}v_nv_jv_1\langle y, v_1^{-1}\mathcal{L}(v_2 \cdots v_{n-1})v_1\rangle v_1^{-1}v_j^{-1}v_n^{-1}v_jv_1 \\ &= v_1^{-1}v_j^{-1}v_nv_jv_1\langle y, \mathcal{L}(v_1 \cdots v_{n-1})\rangle v_1^{-1}v_j^{-1}v_n^{-1}v_jv_1,\end{aligned}$$

where $[v_j^{-1}v_nv_j, v_1yv_1^{-1}] = 1$ either because of relation (3.3) for y, v_1, v_n if $j > 3$, or because of relation (3.5) if $j = 3$. □

If $T' \subseteq T$ is a spanning subgraph (namely, with the same set of vertices), then it is natural to compare the abstract group $\mathrm{A}_{\mathrm{Y}}(T')$ to the "parabolic" subgroup $\langle T'\rangle$ of $\mathrm{A}_{\mathrm{Y}}(T)$, generated by the edges $u \in T'$.

Theorem 3.11. *Let $T' \subseteq T$ be a connected spanning subgraph. Then $\mathrm{A}_{\mathrm{Y}}(T')$ is a retract subgroup of $\mathrm{A}_{\mathrm{Y}}(T)$.*

Proof. Let $\phi\colon \mathrm{A}_{\mathrm{Y}}(T') \to \mathrm{A}_{\mathrm{Y}}(T)$ denote the map defined by $\phi(u) = u$ for $u \in T'$. We construct a map $\psi\colon \mathrm{A}_{\mathrm{Y}}(T) \to \mathrm{A}_{\mathrm{Y}}(T')$ such that $\psi\phi = 1$. It is enough to assume $T = T' \cup \{x\}$. Let β and γ denote the endpoints of x. Since T' is connected (and has the same set of vertices as T), there is a path $v_1, \dots, v_n$ running from β to γ; moreover T being a planar graph, we may assume $x, v_1, \dots, v_n$ is the boundary of a bounded domain in the complement of T, so that $v_1, \dots, v_n$ is a planar path. Let $\hat{x} = \mathcal{L}(v_1 \cdots v_n) \in \mathrm{A}_{\mathrm{Y}}(T')$.

Define $\psi\colon \mathrm{A}_{\mathrm{Y}}(T) \to \mathrm{A}_{\mathrm{Y}}(T')$ by $\psi(u) = u$ for every $u \neq x$ and $\psi(x) = \hat{x}$. It remains to show that ψ is well defined, since clearly $\psi\phi = 1$. For that we need to verify that under the action of ψ, all the relations in $\mathrm{A}_{\mathrm{Y}}(T)$ become trivial in $\mathrm{A}_{\mathrm{Y}}(T')$. If a relation does not involve x, the claim is trivial. Relations of type (3.1) and (3.2) were treated in Lemmas 3.9 and 3.10, respectively. The proof for relations (3.3)–(3.6) is very similar: x is of course one of the edges in the relation, and since $x, v_1, \dots, v_n$ is the boundary of a domain, the other edges must touch x from the outside. The analysis is then very similar to that of the lemmas, and we omit the details. □

By Remark 3.4 we have the following special case.

Corollary 3.12. *Let $T_0 \subseteq T$ be a spanning sub-tree. Then the subgroup $\langle T_0 \rangle$ generated by the edges of T_0 is isomorphic to the braid group.*

4. Action on the Disk

In this section we describe the classical action of the braid group on the fundamental group of an n-punctured disk.

Let D denote the unit disk in $\mathbb{C}$, and $P \subseteq D$ be a subset of n points in the interior of D. For convenience, we will assume the points $P = \{p_1, \dots, p_n\}$ are on the real line, in this order. Obviously $\pi_1(D - P)$ is the free group on $n = |P|$ generators. Consider the group of diffeomorphisms $B = \mathrm{Diff}^+(D - P, \partial D)$. Let us define special elements in this group, denoted by σ_i (for $i = 1, \dots, n-1$), as follows. For a sufficiently small $\epsilon > 0$, let N be the union of ϵ-disks with centers in the line from p_i to p_{i+1}, and let N' be the similar union of 2ϵ-disks. The action of σ_i is to rotate the boundary of N half a circle counterclockwise, exchanging the positions of p_i and p_{i+1}, while preserving all the points outside of N'. It can be easily checked that these elements satisfy the defining relations of the braid group, namely $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and in fact B is generated by the σ_i and defined by these relations, and thus is isomorphic to Artin's braid group. Hence we denote B by B_n. Recall that when multiplying elements of B_n, we compose diffeomorphisms *from left to right*, namely $(\sigma\tau)(u) = \tau(\sigma(u))$ for $\sigma, \tau \in B_n$ and u a point or a path in D.

By a *good path* we mean the *oriented image* of a smooth injective $\gamma\colon (0,1) \to D - P$ which can be extended to a smooth $\gamma\colon [0,1] \to D$ with $\gamma(0) = p_i$ and $\gamma(1) = p_j$

for some $p_i \neq p_j \in P$. Such a path gives rise to an element of B_n in the manner described above (by setting two neighborhoods around γ). This element, which is independent of the orientation only depends on the image of γ, is called the *half-twist* induced by γ, and will be denoted by $\pi(\gamma)$. We will usually omit π. For example, σ_i is nothing but $\pi(h)$ for h the straight line from p_i to p_{i+1}; by abuse of notation, we will also denote this line by σ_i.

Definition 4.1. Let $\vec{\Psi}$ denote the set of good paths in D, up to continuous deformation, and Ψ denote the set obtained from it by forgetting orientation.

Thus we have defined above a map $\pi : \Psi \to B_n$. On the other hand, B_n acts on $\vec{\Psi}$, which induces an action of Ψ and $\vec{\Psi}$ on $\vec{\Psi}$ and Ψ, respectively. Let $\gamma, \delta \in \vec{\Psi}$. If (the closures of) γ and δ do not intersect, then clearly $(\pi(\delta))(\gamma) = \delta(\gamma) = \gamma$. Now assume that γ has endpoints p_i and p_j, and δ has endpoints p_j and p_k $(i, j, k$ distinct). By definition, $(\pi(\delta))(\gamma) = \delta(\gamma)$ is the (good) path going from p_i along γ until coming close to p_j, then circling p_j counterclockwise, and continuing with δ to its endpoint p_k; see Fig. 9.

Now let $\theta \in B_n$ be arbitrary, and let $\gamma, \delta \in \vec{\Psi}$. Consider the good path $\theta(\gamma)$ together with its neighborhoods $N \subseteq N'$. A point outside N' is not moved by either $\theta^{-1}\pi(\gamma)\theta$ or $\pi(\theta(\gamma))$. On the other hand, γ is rotated half a circle counterclockwise under both actions, and so

$$\pi(\theta(\gamma)) = \theta^{-1}\pi(\gamma)\theta. \tag{4.1}$$

Acting with this element on $\theta(\delta)$ we get

$$(\pi(\theta(\gamma)))(\theta(\delta)) = (\theta^{-1}\pi(\gamma)\theta)(\theta(\delta)) = (\pi(\gamma)\theta)(\delta) = \theta(\pi(\gamma)(\delta));$$

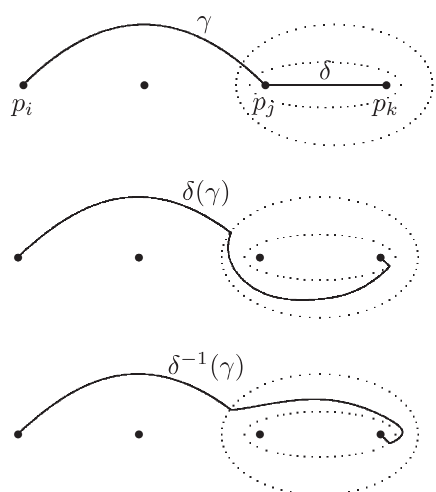

Fig. 9. The action of $\pi(\delta)$ and $\pi(\delta)^{-1}$ on the good path γ.

we record this distributivity law as

$$\theta(\pi(\gamma)(\delta)) = \pi(\theta(\gamma))(\theta(\delta)). \tag{4.2}$$

Corollary 4.2. *For any* $\gamma, \delta \in \vec{\Psi}$,

$$\pi(\pi(\delta)(\gamma)) = \pi(\delta)^{-1}\pi(\gamma)\pi(\delta). \tag{4.3}$$

Proof. Take $\theta = \pi(\delta)$ in Eq. (4.1). □

A sequence of good paths $\gamma_1, \ldots, \gamma_m$ is called a *partial frame* if every γ_i shares exactly one endpoint with γ_{i+1}, and there are no other intersection points. If $m = n-1$, this is called a *frame*. In particular, $\sigma_1, \ldots, \sigma_{n-1}$ is called the *standard frame*.

Remark 4.3. B_n acts transitively on partial frames of any given length, and in particular on frames.

Proof. There is a diffeomorphism taking any given non-self-intersecting path to any other non-self-intersecting path, and the union of paths composing a frame is non-self-intersecting. □

We will need the following refinement of this remark. We say that a good path is *simple* if it does not intersect the real line except for the two endpoints.

Proposition 4.4. *Any good path* $\omega \in \Psi$ (*see Definition* 4.1) *can be written in the form* $\pi(\gamma_1)^{\pm 1} \ldots \pi(\gamma_s)^{\pm 1}(\gamma_0)$ *for suitable simple paths* $\gamma_0, \gamma_1, \ldots, \gamma_s$, *such that each* γ_i *intersects only with* γ_{i+1} *and* γ_{i-1}.

Proof. By induction on the number of intersections of ω with the real line. Suppose ω begins at some p_i and travels first above the real line. Let p_j denote the point of P farthest from p_i in the segment from p_i to the first intersection of ω with the real line. Take γ_1 to be the simple path going above the real line from p_i to p_j, and ω' is the path starting from p_j following ω. Then $\omega = \pi(\gamma_1)(\omega')$ if $i < j$ and $\omega = \pi(\gamma_1)^{-1}(\omega')$ if $i > j$. If $j = i$, deform ω so that it first travels below the real line. Likewise, if ω first travels below the real line, then $\omega = \pi(\gamma_1)^{-1}(\omega')$ if $i < j$ and $\omega = \pi(\gamma_1)(\omega')$ if $i > j$. The proof is complete since ω' has less real points than ω. □

In connection with geometric actions on the disk, we will need the following well known fact.

Remark 4.5. Let $n \geq 4$. The centralizer of σ_1 in B_n is generated by $\sigma_1, \sigma_3, \ldots, \sigma_{n-1}$ and the half-twist $\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^2\sigma_2$ (see Fig. 10). Conjugating, we obtain the centralizer of an arbitrary half-twist.

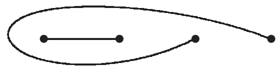

Fig. 10. $\pi(\sigma_1)$ and $\pi(\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^2\sigma_2)$.

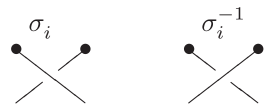

Fig. 11. The braids associated to σ_i and σ_i^{-1} (going downwards).

Recall the standard description of B_n as the group of braids on n strands, where σ_i is viewed as exchanging strands i and $i+1$ with i going above $i+1$, as in Fig. 11.

More generally, if $\gamma \in \Psi$ has endpoints p_i, p_j, then $\pi(\gamma)$ can be realized as the braid obtained by traveling with strands i and j halfway along γ, going beyond strands k whenever γ is above p_k, and above the strand when γ is below p_k; when the strands i and j meet, they are exchanged with the lower index strand going above the higher index one. For example, see Fig. 12.

Example 4.6. It is easy to see that $\sigma_2(\sigma_1)$ is the path connecting p_1 and p_3 and going under p_2. Computing in the braid group, this amounts to an exchange of strands 1 and 3 going *above* strand 2. By Eq. (4.3), we know that $\pi(\sigma_2)(\sigma_1) = \sigma_2^{-1}\sigma_1\sigma_2$, as illustrated in Fig. 13.

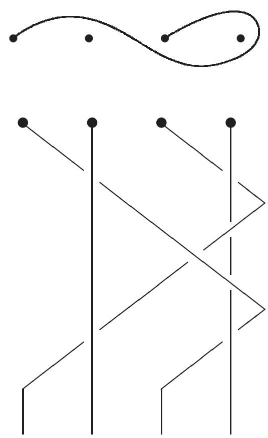

Fig. 12. A good path in the unit disk, and the induced braid.

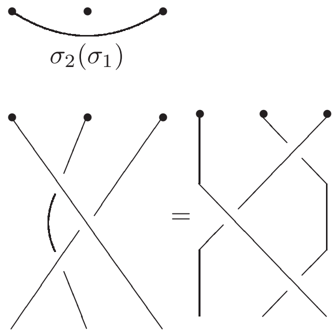

Fig. 13. $\pi(\pi(\sigma_2)(\sigma_1))$ versus the product $\pi(\sigma_2)^{-1}\pi(\sigma_1)\pi(\sigma_2)$.

5. Geometric Extensions and Actions

5.1. *Geometric extensions*

Let S be an arbitrary group, acting transitively on a space Ω. Let $\alpha \in \Omega$. We denote by $S * \Omega$ the HNN extension [7] of S with respect to the identity map on the stabilizer $\mathrm{Stab}(\alpha) \leq S$. Namely, $S * \Omega$ is the group generated by S and an element x, subject to the relations

$$\text{(C)} \qquad \text{if } \sigma(\alpha) = \alpha, \quad \text{then } \sigma x = x\sigma.$$

As all stabilizers are conjugate, different choices of α yield isomorphic groups. Moreover, the action of S extends to an action of $S * \Omega$ on Ω, by letting x act trivially.

If the projection $S * \Omega \to S$ defined by $x \mapsto 1$ factors through a group G, then $G = \langle S, x\rangle$ and (C) holds. We call such a group a *geometric extension* of S.

Example 5.1. (1) When S acts sharply transitively on Ω, the condition (C) trivially holds. In this case $S * \Omega$ is the free product $S * \mathbb{Z}$.

(2) If $\Omega = \{\bullet\}$ is a singleton, then $S * \Omega = S \times \mathbb{Z}$. In this case every other geometric extensions has the form $S \times \mathbb{Z}_n$ for some $n \in \mathbb{Z}$.

(3) More generally if $S_0 \leq S$ is a subgroup and S acts by left multiplication on the quotient space $\Omega = S/S_0$, then $S * \Omega$ is the HNN extension of S with respect to the identity map on S_0.

Recall that a monomorphism $\iota : S \to G$ is a *retraction* if there is an epimorphism $\psi : G \to S$ such that $\psi \circ \iota = 1_S$. Then S is called a *retract subgroup* of G. Letting $\iota : S \to S * \Omega$ be the inclusion map, $\epsilon : S * \Omega \to S$ defined by $\epsilon|_S = 1_S$ and $\epsilon(x) = 1$ satisfies $\epsilon \circ \iota = 1_S$, so S is a retract of $G = S * \Omega$.

Let $\iota : S \to H$ be a retraction with the epimorphism $\epsilon : H \to S$ satisfying $\epsilon \circ \iota = 1_S$. Suppose H is generated by S and an element x, and suppose that the natural epimorphism from the free product $S * \langle x\rangle$ to S, defined by $x \mapsto 1$, factors through H.

Clearly $K = \mathrm{Ker}(\epsilon)$ is the normal subgroup of H generated by x, and $H = SK$. It is also easy to see that $K = \langle x \rangle^S$, the subgroup generated by all the conjugates $\{\sigma x \sigma^{-1} : \sigma \in S\}$. As hinted in Sec. 1, our aim in this paper is to find a good description of K in a certain geometric setting. Since S will be large, the set $\{\sigma x \sigma^{-1} : \sigma \in S\}$ is too large for this purpose.

5.2. *Geometric actions*

Definition 5.2. Let H be a group with a subgroup S acting transitively on a space Ω and let $\alpha \in \Omega$ and $x \in G$ be distinguished elements. We say that the system $(H, S, \Omega, x, \alpha)$ is a *geometric action*, if the embedding $S \hookrightarrow H$ is a retraction, $H = \langle S, x \rangle$, and the condition (C) holds.

This is the case if and only if the epimorphism $\epsilon_1 : S * \Omega \to S$ factors through $\epsilon : H \to S$, so in particular H is a geometric extension of S.

Let $(H, S, \Omega, x, \alpha)$ be a fixed geometric action. For $\omega \in \Omega$, let $\sigma \in S$ be an element such that $\sigma(\omega) = \alpha$. We claim that

$$x_\omega = \sigma x \sigma^{-1} \tag{5.1}$$

is a well-defined element of K, i.e. independent of the choice of σ. Indeed, if $\sigma'(\alpha) = \sigma(\alpha) = \omega$, then $\sigma^{-1}\sigma'(\alpha) = \alpha$ and so $\sigma^{-1}\sigma'$ commutes with x by the condition (C); recall that we compose functions from the left. Notice that $x_\alpha = x$.

Remark 5.3. For every $\omega \in \Omega$ and $\tau \in S$, we have that

$$\tau^{-1} x_\omega \tau = x_{\tau(\omega)}. \tag{5.2}$$

Corollary 5.4. *The kernel $K = \ker(\epsilon : H \to S)$ is generated* (*rather than normally generated*) *by the elements $\{x_\omega : \omega \in \Omega\}$.*

The notion of geometric action can easily be extended to a system $(H, S, \Omega, X, \{\alpha_x\}_{x \in X})$ where: $S \leq H$ is a retract subgroup acting transitively on Ω; $X \subseteq H$ is a (usually finite) subset such that $H = \langle S, X \rangle$; there is an epimorphism $\epsilon : H \to S$ defined by $\sigma \mapsto \sigma$ for $\sigma \in S$ and $x \mapsto 1$ for every $x \in X$; and for every $x \in X$ there is a fixed element $\alpha_x \in \Omega$ such that condition (C) holds. No extra relation is assumed among the generators in X.

5.3. *Maximal geometric quotients*

In order to handle more general groups in terms of geometric actions, we make the following definition.

Definition 5.5. Let $(H, S, \Omega, X, \{\alpha_x\}_{x \in X})$ be a geometric action.

We define $G = \mathcal{G}(H, S, \Omega, X, \{\alpha_x\}_{x \in X})$ as the quotient group of H with respect to the normal subgroup generated by the commutators

$$\{[\sigma, x] : x \in X,\ \sigma \in \mathrm{Stab}(\alpha_x) \subseteq S\}.$$

Finally, let $\mathcal{K}(H, S, \Omega, X, \{\alpha_x\}_{x \in X})$ denote the kernel of the epimorphism $G \to S$.

Clearly $G = \mathcal{G}(H, S, \Omega, X, \{\alpha_x\})$ is the largest quotient of H which is a geometric extension of S (with respect to the given action on Ω).

Similarly to Example 5.1, we have the following example.

Example 5.6. Let $(H, S, \Omega, X, \{\alpha_x\})$ be a geometric action.

(1) If S acts sharply transitively on Ω, then the stabilizers are trivial, and so $\mathcal{G}(H, S, \Omega, X, \{\alpha_x\}) = H$.
(2) If Ω is a singleton then $\mathcal{G}(H, S, \Omega, X, \{\alpha_x\}) \cong S \times \langle X \rangle$, where $\langle X \rangle$ is the subgroup of H generated by X.
(3) Let $\sim$ be an equivalence relation on Ω, consistent with the action of S. Then there is a well-defined action of S on $\Omega/{\sim}$, and $G' = \mathcal{G}(H, S, \Omega/{\sim}, X, \{[\alpha_x]\})$ is a quotient of $G = \mathcal{G}(H, S, \Omega, X, \{\alpha_x\})$. For every $x \in X$, let Γ_x be a set of elements of S such that for every $\beta \sim \alpha_x$, $\tau(\alpha_x) = \beta$ for some $\tau \in \Gamma_x$. Then $\mathrm{Stab}([\alpha_x]) = \mathrm{Stab}(\alpha_x) \cdot \Gamma_x$. Therefore, the kernel of the epimorphism $G \to G'$ is generated, as a normal subgroup, by the set of commutators $\{[\tau, x] : x \in X,\ \tau \in \Gamma_x\}$.

Proposition 5.7. *Notation as in Definition 5.5, let G be the group $\mathcal{G}(H, S, \Omega, X, \{\alpha_x\})$. Then, the system*

$$(G, S, \Omega, X, \{\alpha_x\})$$

is a geometric action. In particular, we have that $G = K \ltimes S$ where $K = \mathcal{K}(H, S, \Omega, X, \{\alpha_x\})$ is the kernel, and the action is given in (5.2).

Proof. It remains to show that the map $S \to G$ defined by projecting $\sigma \in S \leq H$ modulo the relations is a retraction. Let $\epsilon : H \to S$ be the epimorphism defined by $\epsilon(x) = 1$ for $x \in X$, and let $\iota : S \to H$ be the embedding, so that $\epsilon \circ \iota = 1_S$. Let $\psi : H \to G$ be the natural epimorphism.

The map $\epsilon' : G \to S$ defined by $\sigma \mapsto \sigma$ and $x \mapsto 1$ preserves the commutation relations and so is well defined, and $\epsilon' \circ \psi = \epsilon$. Let $\iota' = \psi \circ \iota$. Then $\epsilon' \circ \iota' = \epsilon' \circ \psi \circ \iota = \epsilon \circ \iota = 1_S$. ◻

We will later need ways of comparing two geometric extensions.

Lemma 5.8. *Let $(H, S, \Omega, X, \{\alpha_x\})$ and $(H', S', \Omega, Y, \{\beta_y\})$ be two geometric actions.*

Let $\psi : H \to H'$ be an isomorphism, inducing an isomorphism $S \to S'$ which commutes with the action (namely $\psi(\sigma)(\omega) = \sigma(\omega)$ for $\omega \in \Omega$). Suppose that for every $x \in X$ there are $\tau \in S'$ and $y \in Y$ such that $\psi(x) = \tau^{-1} y \tau$ and $\beta_y = \tau^{-1}(\alpha_x)$. Then ψ induces an isomorphism

$$G = \mathcal{G}(H, S, \Omega, X, \{\alpha_x\}) \to G' = \mathcal{G}(H', S', \Omega, Y, \{\beta_y\}).$$

Proof. Since G is defined as the quotient of H with respect to the relations $[\sigma, x] = 1$ for every $x \in X$ and $\sigma \in \mathrm{Stab}(x)$, and G' is defined similarly as a quotient of H', it is enough to prove that ψ transfers such relations to suitable relations in G'.

Let $x \in X$, and let $\sigma \in S$ be an element such that $\sigma(\alpha_x) = \alpha_x$. Write $\psi(x) = \tau^{-1} y \tau$ for $y \in Y$ and $\tau \in S'$, and $\beta_y = \tau^{-1}(\alpha_x)$. Then $(\tau\psi(\sigma)\tau^{-1})(\beta_y) = (\tau\psi(\sigma)\tau^{-1})(\tau^{-1}(\alpha_x)) = \tau^{-1}(\psi(\sigma)(\alpha_x)) = \tau^{-1}(\sigma(\alpha_x)) = \tau^{-1}(\alpha_x) = \beta_y$ (acting from left to right), and so

$$[\psi(\sigma), \psi(x)] = [\psi(\sigma), \tau^{-1} y \tau] = [\tau\psi(\sigma)\tau^{-1}, y] = 1. \qquad \square$$

6. The Case of a Single Cycle

Consider the graph $T^{(1)}$ composed of the standard frame and one undirected path u connecting p_1 and p_n from above, as in Fig. 14. We let $\vec{\alpha}$ denote the path u directed from p_1 to p_n.

We assume $n \geq 4$. By definition, $\mathrm{A}(T^{(1)})$ is generated by elements corresponding to the paths $\sigma_1, \ldots, \sigma_{n-1}$ and u, and we use this notation for the generators as well. By Theorem 3.11, the subgroup $\langle \sigma_1, \ldots, \sigma_{n-1} \rangle$ of $\mathrm{A}(T)$ satisfies only the braid relations, and we denote this subgroup by B_n.

Let α denote the element

$$\alpha = \mathcal{L}(\sigma_1 \cdots \sigma_{n-1}) = \sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_{n-2}^{-1} \sigma_{n-1} \sigma_{n-2} \ldots \sigma_2 \sigma_1 \in B_n. \tag{6.1}$$

We have that $\pi(\vec{\alpha}) = \alpha \in B_n \subseteq \mathrm{A}(T^{(1)})$. Set $x = u\alpha^{-1}$. Since $\mathrm{A}(T^{(1)})$ is generated by B_n and u, we also have that $\mathrm{A}(T^{(1)}) = \langle B_n, x \rangle$, where, by the proof of Theorem 3.11, we have an epimorphism $\mathrm{A}(T^{(1)}) \to B_n$ which is the identity on B_n and sends $x \mapsto 1$.

Recall that by Remark 4.3, B_n acts transitively on $\vec{\Psi}$ (defined in Definition 4.1). In this section we study the largest quotient of $\mathrm{A}(T^{(1)})$ which acts geometrically on $\vec{\Psi}$, namely the group

$$G = \mathcal{G}(\mathrm{A}(T^{(1)}), B_n, \vec{\Psi}, x, \vec{\alpha}) \tag{6.2}$$

of Definition 5.5. Notice that for the current graph $T^{(1)}$, $\mathrm{A}(T^{(1)}) = \mathrm{A}_{\mathrm{Y}}(T^{(1)})$.

Remark 6.1. By Definition 3.1, $\mathrm{A}(T^{(1)})$ is the group generated by u and $\sigma_1, \ldots, \sigma_{n-1}$, with the relations:

$$\begin{aligned} [\sigma_i, \sigma_j] &= 1 && \text{for } |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} && \text{for } i = 1, \ldots, n-2, \\ [u, \sigma_i] &= 1 && \text{for } i = 2, \ldots, n-2, \end{aligned}$$

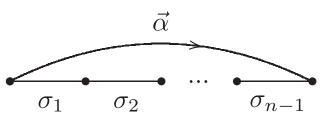

Fig. 14. The graph $T^{(1)}$ of Sec. 6.

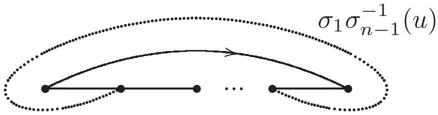

Fig. 15. A path acting trivially on α.

and

$$\sigma_1 u \sigma_1 = u \sigma_1 u, \tag{6.3}$$

$$\sigma_{n-1} u \sigma_{n-1} = u \sigma_{n-1} u. \tag{6.4}$$

Since α is conjugate to σ_1, we can conclude the following from Remark 4.5.

Remark 6.2. The stabilizer of $\vec{\alpha}$ in the action of B_n on $\vec{\Psi}$ is generated by $\sigma_2, \ldots, \sigma_{n-2}$, $\sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}$ (see Fig. 15), and α^2. Note that α reverses the orientation of $\vec{\alpha}$.

Since G is the quotient of $\mathrm{A}(T^{(1)}) = \langle B_n, u\rangle$, obtained by letting $x = u\alpha^{-1}$ commute with $\mathrm{Stab}(\vec{\alpha})$, we have the following summary.

Summary 6.3. *The group $G = \langle B_n, x\rangle$ is defined by the relations*:

$$\sigma_1 x\alpha\sigma_1 = x\alpha\sigma_1 x\alpha, \tag{6.5}$$

$$\sigma_{n-1} x\alpha\sigma_{n-1} = x\alpha\sigma_{n-1} x\alpha, \tag{6.6}$$

$$[x, \sigma_i] = 1 \quad (i = 2, \ldots, n-2), \tag{6.7}$$

$$[x, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1, \tag{6.8}$$

$$[x, \alpha^2] = 1. \tag{6.9}$$

Writing $x_{\vec{\alpha}}$ instead of x, the commutation relations become

$$\sigma_i x_{\vec{\alpha}} = x_{\vec{\alpha}}\sigma_i \quad (1 < i \leq n-2), \tag{6.10}$$

$$\sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1} x_{\vec{\alpha}} = x_{\vec{\alpha}}\sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}, \tag{6.11}$$

$$\alpha^2 x_{\vec{\alpha}} = x_{\vec{\alpha}}\alpha^2, \tag{6.12}$$

while Eqs. (6.5) and (6.6) translate to

$$\sigma_1 x_{\vec{\alpha}}\alpha\sigma_1 = x_{\vec{\alpha}}\alpha\sigma_1 x_{\vec{\alpha}}\alpha, \tag{6.13}$$

$$\sigma_{n-1} x_{\vec{\alpha}}\alpha\sigma_{n-1} = x_{\vec{\alpha}}\alpha\sigma_{n-1} x_{\vec{\alpha}}\alpha. \tag{6.14}$$

Equations (6.10)–(6.12) are equivalent to the assumption on geometric action, in particular condition (C) of Sec. 5, which implies that x_ω of Eq. (5.1), defined as

$\sigma x \sigma^{-1}$ for some $\sigma \in B_n$ such that $\sigma(\omega) = \vec{\alpha}$, is well defined for every $\omega \in \vec{\Psi}$. Relation (6.13) is equivalent to

$$\sigma_1 x_{\vec{\alpha}} \sigma_1^{-1} = x_{\vec{\alpha}} \alpha \sigma_1 x_{\vec{\alpha}} \sigma_1^{-1} \alpha^{-1},$$

and applying Eq. (5.2), this becomes

$$x_{\sigma_1^{-1}(\vec{\alpha})} = x_{\vec{\alpha}} \cdot x_{\alpha^{-1}(\sigma_1^{-1}(\vec{\alpha}))}. \tag{6.15}$$

Likewise relation (6.14) translates to

$$x_{\sigma_{n-1}^{-1}(\vec{\alpha})} = x_{\vec{\alpha}} \cdot x_{\alpha^{-1}(\sigma_{n-1}^{-1}(\vec{\alpha}))}. \tag{6.16}$$

Slightly generalizing Remark 4.3, it is easy to see that B_n acts transitively on oriented triangles whose three vertices are in the set P of the n vertices of $T^{(1)}$, and which contain no other points from P in the interior. The three directed paths $\sigma_1^{-1}(\vec{\alpha}), \vec{\alpha}$ and $\alpha^{-1}(\sigma_1^{-1}(\vec{\alpha}))$ of (6.15) form such a triangle (left-hand side of Fig. 16), and so conjugating by a generic element of B_n, we arrive at the relation

$$x_b = x_a \cdot x_c$$

whenever $a, b, c \in \vec{\Psi}$ are directed paths as in the left-hand side of Fig. 17. In a similar manner, conjugating (6.16) (whose corresponding triangle is depicted in the right-hand side of Fig. 16) we obtain the same relation whenever $a, b, c \in \vec{\Psi}$ are directed paths as in the right-hand side of Fig. 17 (note the reverse order of a and c!).

These two situations can be combined into one equation.

Corollary 6.4. *If* $\overrightarrow{a}, \overrightarrow{c} \in \vec{\Psi}$ *form a partial frame* (*namely they go in the same direction*, *connecting three points*, *but it does not matter which one comes first*),

Fig. 16. The paths of Eqs. (6.15) and (6.16) form two triangles.

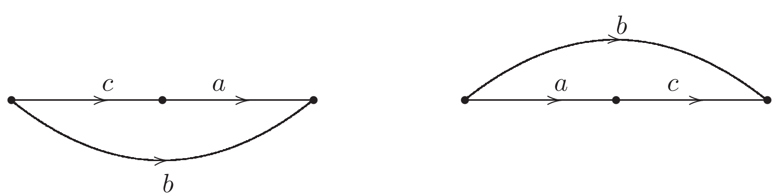

Fig. 17. Relations (6.15) and (6.16).

then

$$x_{a(\overrightarrow{c})} = x_{\overrightarrow{a}} x_{\overrightarrow{c}} \tag{6.17}$$

and

$$x_{a^{-1}(\overrightarrow{c})} = x_{\overrightarrow{c}} x_{\overrightarrow{a}}, \tag{6.18}$$

where a is the good path that obtained by forgetting the direction of $\overrightarrow{a}$.

Since $c(\overrightarrow{a}) = a^{-1}(\overrightarrow{c})$, we can switch a and c in Eq. (6.17) to obtain

$$x_{a^{-1}(\overrightarrow{c})} = x_{\overrightarrow{c}} x_{\overrightarrow{a}}. \tag{6.19}$$

This equation motivates the following notation: if $\overrightarrow{a} \in \vec{\Psi}$ is a directed path and the direction cannot be confused, then we write a instead of $x_{\overrightarrow{a}}$. For example, Fig. 18 demonstrates that if a, c form a partial frame, $\gamma = a(c)$ goes above the frame and $\delta = c(a)$ goes below the frame, then $x_\gamma = x_a x_c$ while $x_\delta = x_c x_a$.

Applying Proposition 4.4, we conclude that every x_ω can be written as a product of the $x_{\overrightarrow{\sigma_i}}$ and $x_{\overleftarrow{\sigma_i}}$.

Corollary 6.5. *The kernel of the epimorphism $G \to B_n$ is generated by the $x_{\overrightarrow{\sigma_i}}$ and $x_{\overleftarrow{\sigma_i}}$ for $i = 1, \ldots, n-1$.*

Let us apply these computations to the triangles in Fig. 19, where for the mean time we denote the path pointing from the endpoint of c to the starting point of a by e. The other two values (namely ce and ea) were computed from the triangles that e forms with c and with a.

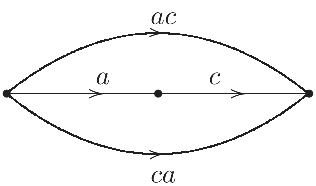

Fig. 18. Basic relations.

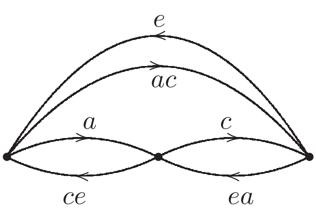

Fig. 19. A complete triangle.

All this was done using the upper triangle in Fig. 18 (going counterclockwise). Going clockwise we obtain

$$ce \cdot ea = e,$$

$$ac \cdot ce = c,$$

$$ea \cdot ac = a.$$

It follows that $e = c^{-2}a^{-1}c$ (which, we recall, is a short notation for $x_e = x_c^{-2}x_a^{-1}x_c$). Let us denote $z = [c, a] = cac^{-1}a^{-1}$. Substituting the value of e in the first equation we get $cc^{-2}a^{-1}cc^{-2}a^{-1}ca = c^{-2}a^{-1}c$, namely $ca^{-1}c^{-1}a^{-1}ca = a^{-1}c$, which is equivalent to $za = az$. The third equation becomes $c^{-2}a^{-1}caac = a$, namely $ca^2c = ac^2a$ or $zac = acz$. Since $za = az$, we obtain $zc = cz$, namely z commutes with both a and c (again, this is a shorthand for "$[x_c, x_a]$ commutes with both x_a and x_c"). It is useful to rewrite Fig. 19 following the recent discoveries: see Fig. 20. We summarize this as follows.

Corollary 6.6. *If $\overrightarrow{a}, \overrightarrow{c} \in \vec{\Psi}$ form a partial frame, and $\overleftarrow{a}, \overleftarrow{c}$ denote the inverse paths, then $x_{\overleftarrow{a}} = zx_{\overrightarrow{a}}^{-1}$ and $x_{\overleftarrow{c}} = zx_{\overrightarrow{c}}^{-1}$, where $z = [x_{\overrightarrow{c}}, x_{\overrightarrow{a}}]$.*

Taking $a = \overrightarrow{\sigma}_i$ and $c = \overrightarrow{\sigma}_{i+1}$ $(i = 1, \ldots, n-2)$, we find that $x_{\overleftarrow{\sigma}_i} = [x_{\overrightarrow{\sigma}_{i+1}}, x_{\overrightarrow{\sigma}_i}]x_{\overrightarrow{\sigma}_i}^{-1}$; similarly for $a = \overrightarrow{\sigma}_{n-1}$ and $c = \overrightarrow{\sigma}_{n-2}$ we have $x_{\overleftarrow{\sigma}_{n-1}} = [x_{\overrightarrow{\sigma}_{n-1}}, x_{\overrightarrow{\sigma}_{n-2}}]x_{\overrightarrow{\sigma}_{n-1}}^{-1}$. We can thus improve Corollary 6.5.

Corollary 6.7. *The kernel K of $G \to B_n$ is generated by the $x_{\overrightarrow{\sigma}_i}$ for $i = 1, \ldots, n-1$.*

Now consider the situation in Fig. 21, where c_1 denotes the inverse path of a, and a_1 denotes the inverse path of c. By the above computation, we see that

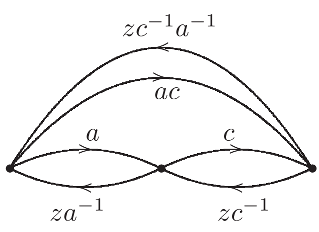

Fig. 20. Figure 19, repeated.

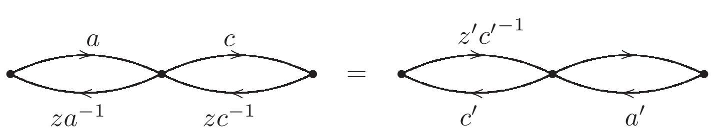

Fig. 21. Proof that commutators have order 2.

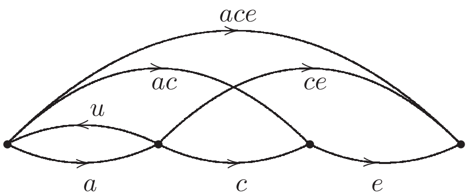

Fig. 22. Proof that $ae = ea$.

the inverse path of a equals za^{-1} (i.e. $x_{\overleftarrow{a}} = zx_{\overrightarrow{a}}$) and the inverse path of c equals zc^{-1}. Likewise, the inverse path of c_1 equals $z_1 c_1{}^{-1}$, where $z = [c, a]$ and $z_1 = [c_1, a_1]$. Comparing, we obtain $a_1 = zc^{-1}$, $c_1 = za^{-1}$ and $a = z_1 c_1{}^{-1}$. Thus $z = c_1 a = ac_1 = z_1$, since a and $c_1 = za^{-1}$ commute. On the other hand, $z_1 = [c_1, a_1] = [za^{-1}, zc^{-1}] = [a^{-1}, c^{-1}] = [c, a]^{-1} = z^{-1}$. So we proved $z^2 = 1$.

Next, consider the diagram in Fig. 22. The values ac and ce are easily computed from the triangles completing a, c and c, e, respectively. In the same manner we obtain the value ace from the triangle of a, ce. From the triangle a, c, ac we compute that $u = [c, a]a^{-1}$. Likewise from the triangle a, ce, ace we get $u = [ce, a]a^{-1}$. Thus $[c, a] = [ce, a]$ and $ae = ea$.

Finally, consider the diagram in Fig. 23. Considering the leftmost triangle, we get from Fig. 20 that $u = [c, a]c^{-1}$. On the other hand for the rightmost triangle, the same argument gives $u = [e, c]c^{-1}$. Therefore $[c, a] = [e, c]$. By Remark 4.3, it follows that as long as a, c form a partial frame, $z = [c, a]$ is independent of a and c.

This situation can be summarized as follows.

Proposition 6.8. *Let $\omega, \omega' \in \vec{\Psi}$ be two directed paths with the same starting and ending points. Then $x_{\omega'} = z(\omega, \omega') \cdot x_\omega$ where $z(\omega, \omega') = 1$ if the number of points of P crossed when ω' is deformed into ω is even, and $z(\omega, \omega') = z$ if this number is odd.*

Proof. By induction on the number of points, and without loss of generality, we may assume that ω is the path denoted as ac in Fig. 18, and ω' is the path denoted ca there. But then $x_\omega = x_a x_c$ and $x_{\omega'} = x_c x_a$, so that $x_\omega x_{\omega'}^{-1} = [x_a, x_c] = z$. $\square$

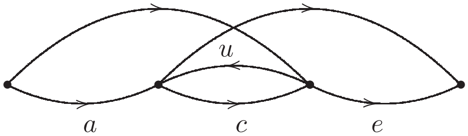

Fig. 23. Proof that all non-trivial commutators in a frame are equal.

Corollary 6.9. *If $\omega \in \vec{\Psi}$ and $\theta \in \Psi$ is a good path such that $\theta(\omega)$ has the same starting and ending points as ω (in particular θ and ω have disjoint endpoints), then $x_{\theta(\omega)} = x_\omega$.*

This follows from Corollary 6.8 by induction on the number of intersection points of θ and ω. For comparison, notice that $x_{\sigma_1(\vec{\sigma_1})} = x_{\overleftarrow{\sigma_1}} = z \cdot x_{\vec{\sigma_1}}$, so the equality does not hold if the starting and ending points are exchanged. Likewise, let $x_{\sigma_1^2(\vec{\sigma_2})} = z \cdot x_{\vec{\sigma_2}}$, and indeed σ_1^2 is not a half-twist.

Definition 6.10. Let K' denote the group generated by $y_1, \ldots, y_{n-1}$ and z, with the defining relations

$$[y_i, y_j] = \begin{cases} 1 & \text{if } |i-j| > 1, \\ z & \text{if } |i-j| = 1, \end{cases}$$
$$z \text{ is central and } z^2 = 1.$$

Thus K' is a central extension of $\mathbb{Z}^{n-1}$ by $\mathbb{Z}/2$.

An action of B_n on K' is defined as follows. For every $1 \le t \le n-1$, $\sigma_t(z) = z$, and

$$\sigma_t(y_r) = \begin{cases} zy_t^{-1} & r = t, \\ y_t y_r & r = t+1, \\ y_r & |r-t| > 1. \end{cases}$$

(This can be verified to be well defined using the braid relations.)

Recall that by Corollary 5.4, G is a semidirect product of $B_n = \{\sigma_1, \ldots, \sigma_{n-1}\}$ and the normal subgroup $K = \langle x_\omega \rangle_{\omega \in \vec{\Psi}}$, with the action given by Eq. (5.2). We can now summarize this section as follows.

Theorem 6.11. *Let $T^{(1)}$ be the graph of Fig.* 14, *and G be the group defined in Eq.* (6.2), *acting geometrically on $\vec{\Psi}$. Then G is a semidirect product of $B_n = \{\sigma_1, \ldots, \sigma_{n-1}\}$ and the normal subgroup $K = \langle x_{\vec{\sigma_1}}, \ldots, x_{\vec{\sigma}_{n-1}} \rangle$, and $K \cong K'$, as B_n-groups* (*see* Definition 6.10), *via the correspondence $x_{\vec{\sigma_i}} \leftrightarrow y_i$. In particular, the generator x maps to the product $y_1 y_2 \cdots y_{n-1}$.*

Notice that since z is the only non-trivial commutator in K', it is in fact central in G. In Theorem 7.1 we realize $G/\langle z \rangle$ as a geometric action on the space of non-directed paths.

7. Geometric Actions on Quotient Spaces

The action of the braid group B_n on the space $\vec{\Psi}$ of directed paths, as given in Sec. 4, naturally induces an action on the space Ψ of non-directed paths. Likewise, there are the obvious actions on the set $\vec{P^2}$ of ordered pairs of points, and on the set P^2 of (non-ordered) pairs of points. These are induced from the natural action of the symmetric group, S_n.

In Sec. 6 (in particular, Theorem 6.11), we computed the group $G(\vec{\Psi}) = \mathcal{G}(\mathrm{A}(T^{(1)}), B_n, \vec{\Psi}, x, \vec{\alpha})$, for $T^{(1)}$ the graph with a single cycle, as in Fig. 14. The purpose of this section is to present the groups obtained in a similar manner for quotients of $\vec{\Psi}$, namely Ψ, $\vec{P^2}$ and P^2. For that, we apply Example 5.6(3).

Let $G(\Psi) = \mathcal{G}(\mathrm{A}(T^{(1)}), B_n, \Psi, x, u)$, where u is the undirected path, whose directed version is $\vec{\alpha}$ in Fig. 14.

Theorem 7.1. *Let $M \subseteq \mathbb{Z}^n$ be the subgroup of zero-sum vectors. Then,*

$$G(\Psi) \cong B_n \ltimes M$$

via the action of B_n induced from that of S_n on M.

Proof. The space Ψ is the quotient of $\vec{\Psi}$ with respect to forgetting direction of arrows.

By Example 5.6(3), $G(\Psi)$ is the quotient of $G(\vec{\Psi})$ with respect to the normal subgroup generated by all the commutators $[x, \gamma]$ for $\gamma \in \Gamma_x$. Here, Γ_x is a subset of B_n *acting transitively* on the equivalence class of $\vec{\alpha}$, which is $\{\vec{\alpha}, \overleftarrow{\alpha}\}$, where $\overleftarrow{\alpha}$ is the directed path $\vec{\alpha}$, reversed. Since $\alpha(\vec{\alpha}) = \overleftarrow{\alpha}$, we may choose $\Gamma_x = \{1, \alpha\}$. It follows that $G(\Psi) = G(\vec{\Psi})/\langle [x, \alpha] \rangle$.

Recall the notation for $x_\omega (\omega \in \vec{\Psi})$ from Sec. 6; in particular the generator x itself is identified as $x = x_{\vec{\alpha}}$. Now, $[\alpha, x] = \alpha x_{\vec{\alpha}} \alpha^{-1} x_{\vec{\alpha}}^{-1} = x_{\overleftarrow{\alpha}} x_{\vec{\alpha}}^{-1} = z$, by (5.2) and Corollary 6.6.

From Definition 6.10 it is clear that $K'/\langle z \rangle \cong M$ as B_n-sets, so we are done by Theorem 6.11. ◻

Next, we compute $G(\vec{P^2}) = \mathcal{G}(\mathrm{A}(T^{(1)}), B_n, \vec{P^2}, x, u)$. Set $p = (1, n)$, the ordered pair of endpoints of α. If $\omega \in B_n$, then obviously $\omega^2(p) = p$, so $\langle \omega^2 \rangle \subseteq \mathrm{Stab}(p)$. On the other hand, since B_n modulo squares equals S_n, and the stabilizer of p under the action of S_n is the symmetric group $S_{2,\ldots,n-1}$, we clearly have that $\mathrm{Stab}(p) = \langle \omega^2, \sigma_2, \ldots, \sigma_{n-2} : \omega \in B_n \rangle$.

Corollary 7.2. $G(\vec{P^2}) = G(\Psi)$.

Proof. $G(\vec{P^2})$ is obtained from $G(\Psi)$ by taking the quotient with respect to the commutators $[x, \tau]$ for every $\tau \in \mathrm{Stab}(p)$. On the one hand, x commutes with $\sigma_2, \ldots, \sigma_{n-2}$ already in $G(\Psi)$. On the other hand, the action of ω^2 on $\vec{\alpha}$ does not change the endpoints, and so $[\omega^2, x] = \omega^2 x_{\vec{\alpha}} \omega^{-2} x_{\vec{\alpha}}^{-1} = x_{\omega^2(\vec{\alpha})} x_{\vec{\alpha}}^{-1}$, which is either 1 or z by Proposition 6.8. By the same proposition, $[\sigma_1^2, x] = z$, and therefore $\langle [x, \tau] : \tau \in \mathrm{Stab}(p) \rangle = \langle z \rangle$. It follows that $G(\vec{P^2}) = G(\Psi)/\langle z \rangle$ which is $G(P^2)$ by the previous theorem. ◻

Clearly $\mathrm{Stab}(\{1,n\}) = \mathrm{Stab}(p) \cdot \mathrm{Stab}(\alpha)$, where $p = (1,n)$ as above, and α is the undirected path as above. From the above corollary we immediately conclude for $G(P^2) = \mathcal{G}(\mathrm{A}(T^{(1)}), B_n, P^2, x, u)$ the following corollary.

Corollary 7.3. $G(P^2) = G(\vec{P^2}) = G(\Psi)$.

8. Reduction from Geometry to Combinatorics

Let T be a planar graph, and let x, y be edges with one common vertex. We say that x, y, x' form a *virtual triangle* in T if the other two vertices of x and y are disjoint; x' is an edge connecting them; x' does not intersect any edge of T; and the triangle bounded by x, y, x' does not contain any vertex of T.

If x, y, x' are edges of a virtual triangle in T, which are ordered counterclockwise, as in Fig. 24, we say that they form an *ordered virtual triangle*. We denote the respective vertices by p, q, r (which are distinct by assumption).

Consider the categories of connected graphs and groups, with the standard morphisms. The functors we consider here are maps from graphs to groups, sending morphisms to morphisms. A functor F sends the edges of a graph T to elements in the group $F(T)$. We say that the functor is *tight*, if $F(T)$ is generated by (the images under F of) the edges of T. Thus A and A_{Y} are examples of tight functors.

A tight functor has the *triangulation property* if, for every two planar graphs T and T' such that T' is obtained from T by deleting an edge x and inserting an edge x', where x, y, x' is an ordered virtual triangle with $y \in T \cap T'$, the map defined by $t \mapsto t$ for $x' \neq t \in T'$ and $x' \mapsto x^{-1}yx$ defines an isomorphism $F(T') \to F(T)$. Equivalently, $t \mapsto t$ for $x \neq t \in T$ and $x \mapsto y^{-1}x'y$ defines an isomorphism $F(T) \to F(T')$. (The direction is always counterclockwise: the first edge is mapped to the result of the second acting on the third.)

Theorem 8.1. *The functor* A_{Y} *has the triangulation property.*

Proof. Let T'' denote the union $T \cup T'$. By Theorem 3.11, there are well defined maps ϕ, ϕ', ψ, ψ' as in Fig. 25, while $\psi\phi$ and $\psi'\phi'$ are the identity maps on the groups $\mathrm{A}_{\mathrm{Y}}(T)$ and $\mathrm{A}_{\mathrm{Y}}(T')$, respectively.

By their definitions, given in the proof, the composition $\psi\phi'$ sends $u \mapsto u$ for every $u \neq x, x'$ in T'', and $\psi\phi'(x') = \psi(x') = \mathcal{L}(xy) = x \cdot y = x^{-1}yx$. Likewise

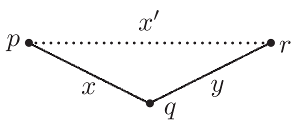

Fig. 24.

$$\begin{tikzcd}
& \mathrm{A}_{\mathrm{Y}}(T'') \arrow[dl, shift left, "\psi'"] \arrow[dr, shift left, "\psi"] & \\
\mathrm{A}_{\mathrm{Y}}(T') \arrow[ur, hook, shift left, "\phi'"] & & \mathrm{A}_{\mathrm{Y}}(T) \arrow[ul, hook', shift left, "\phi"']
\end{tikzcd}$$

Fig. 25.

$\psi'\phi$ sends $u \mapsto u$ for $u \neq x, x'$, and $\psi'\phi(x) = y^{-1}x'y$. It follows that $(\psi'\phi)(\psi\phi')$ acts as the identity on every generator $u \neq x'$ in $\mathrm{A}_{\mathrm{Y}}(T')$. Moreover, $(\psi'\phi\psi\phi')(x') = (\psi'\phi)(x \cdot y) = (\psi'\phi)(x) \cdot (\psi'\phi)(y) = (y \cdot x') \cdot y = y^{-1}x'^{-1}yyy^{-1}x'y = y^{-1}x'^{-1}yx'y = y^{-1}x'^{-1}x'yx' = x'$, so $\psi'\phi\psi\phi'$ is the identity on $\mathrm{A}_{\mathrm{Y}}(T')$. Likewise $\psi\phi'\psi'\phi$ is the identity on $\mathrm{A}_{\mathrm{Y}}(T)$, proving that $\psi'\phi$ is the required isomorphism. □

Remark 8.2. Let $\theta \in B_n$ be an arbitrary element, and let $T' = \theta(T)$ be the graph obtained from a planar graph T by the action of θ on the edges, as in Sec. 4. Then $\mathrm{A}_{\mathrm{Y}}(T) \cong \mathrm{A}_{\mathrm{Y}}(T')$ (indeed, $\mathrm{A}_{\mathrm{Y}}(T)$ is defined abstractly, depending only on the isomorphism class of T as a planar graph).

Let $\mathcal{T}_n$ denote the set of connected planar graphs on the vertices $\{1, \ldots, n\}$. We say that $T, T' \in \mathcal{T}_n$ are equivalent, if, for suitable edges $x, y, z \in T \cup T'$ which form a minimal triangle (namely, a triangle with no vertices in the interior), we have that $T - T' = \{y\}$, $T' - T = \{z\}$. We can now define an equivalence relation on $\mathcal{T}_n$, by allowing sequences of triangular steps. Theorem 8.1 provides, for equivalent graphs T and T', the isomorphism $\mathrm{A}_{\mathrm{Y}}(T) \cong \mathrm{A}_{\mathrm{Y}}(T')$.

Theorem 8.3. *Every two connected planar graphs on the set of vertices* $\{1, \ldots, n\}$, *with the same number of edges, are equivalent.*

Proof. We first claim that T is equivalent to a "fat tree", namely a graph whose minimal cycles all have (graph) length 2, corresponding to multiple edges. We induct on the number of (bounded) connected components of the complement of T in a fixed disk, for which the length of the boundary is more than 2. This number is zero if and only if T is a fat tree.

Let $v_0, \ldots, v_n$ denote the edges composing the boundary of a component D, enumerated counterclockwise, as in Fig. 8. Notice that although the same edge may be present twice in this list, every v_i has a well defined starting point, which we denote by p_i. Clearly p_0 is the endpoint of v_n. Let i be minimal with respect to the property that $p_i, p_{i+1}, \ldots, p_{n-1} \neq p_0$. Clearly $0 < i \leq n-1$. But since $p_{i-1} = p_0$ is the endpoint of v_n, $v_{i-1}, \ldots, v_n$ form a cycle in the graph, so we may assume $i = 1$, namely p_0 is not on any of $v_1, \ldots, v_{n-1}$.

By successive triangulation, we can now replace v_n by $v'_{n-1} = v_{n-1} \cdot v_n$ (see Definition 3.7), then v'_{n-1} by $v'_{n-2} = v_{n-2} \cdot (v_{n-1} \cdot v_n)$, etc., until v'_2 is replaced by

$v'_1 = v_1 \cdot (v_2 \cdots (v_{n-1} \cdot v_n) \cdots)$, which has the same endpoints as v_0. This process defines a sequence of graphs $T_{(n-1)}, \ldots, T_{(1)}$ equivalent to $T = T_{(n)}$, where the cycle $v_0, \ldots, v_n$ of T is replaced by $v_0, \ldots, v_{i-1}, v'_i$ in $T_{(i)}$. Passing from $T_{(n)}$ to $T_{(1)}$ reduces the number of domains with long boundaries in the complement, as asserted.

Now suppose T is a fat tree. Let $p_{i_1}, \ldots, p_{i_m}$ be the vertices connected by a partial frame of maximal length, and assume $m < n$. Since T is connected, there is some p_r ($r \neq i_1, \ldots, i_m$) connected by an edge to some p_{i_k} ($1 < k < m$). Choose the pair p_r, p_{i_k} with k maximal. Via triangulation, the edge connecting p_r and p_{i_k} can be replaced by an edge connection p_r and $p_{i_{k+1}}$, giving rise to an equivalent graph; this can be repeated for all the parallel edges, in case the tree was fat at this edge. Inducting on k we eventually get a longer partial frame. Inducting on the length of the partial frame, we may eventually assume T is a fat path.

Finally, triangulating further, we can collect all multiple edges to a fat edge connecting the endpoints of the frame, resulting (up to isomorphism of planar graphs) in the (undirected) graph of Fig. 28; and this graph only depends on n and the original number of edges. $\square$

Corollary 8.4. *Let T be a planar graph on n vertices. Then* $\mathrm{A_Y}(T)$ *depends only on n and the first homology of T.*

Theorem 8.1 can be used to give a functorial interpretation to the relations defining the groups $\mathrm{A_Y}(T)$. For tight functors F_1 and F_2, we say that F_1 is larger than F_2 if, for every graph T, the map from $F_1(T)$ to $F_2(T)$, sending $F_1(u)$ to $F_2(u)$ for every $u \in T$, is onto. Our main interest is in functors smaller than A (of Definition 3.1).

Theorem 8.5. *The functor* $\mathrm{A_Y}$ (*defined in Definition* 3.2) *is the largest with the triangulation property among all the tight functors smaller than* A.

Proof. The proof is similar in spirit to that of Theorem 3.8. By Theorem 8.1, all we need to show is that the triangulation property (for quotients of $A(T)$) implies the relations of Definition 3.2.

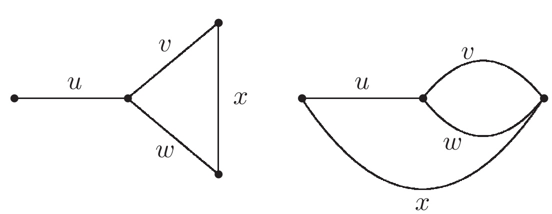

Fig. 26. T_1 and T_2.

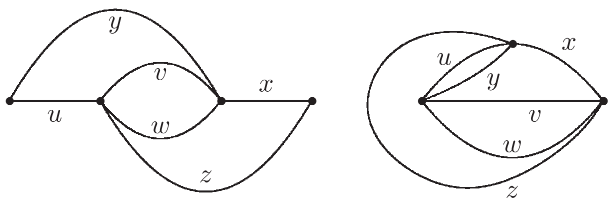

Fig. 27. T_3 and T_4.

Let T_1 denote the graph on the left of Fig. 26, and consider its subgraphs $\{u, w, x\}$ and $\{u, v, w\}$. In $\mathrm{A}(\{u, w, x\})$ we have the relation $[u, x] = 1$, so the triangulation property provides $[u, v^{-1}wv] = 1$ in the group associated to $\{u, v, w\}$. This is relation (3.3).

Next, consider the graph T_2, on the right-hand side of Fig. 26. Since $\langle x, v\rangle = 1$ in $\mathrm{A}(\{x, v, u\})$, we obtain the relation $\langle w^{-1}uw, v\rangle = 1$ is $\mathrm{A}(\{u, v, w\})$, which is one case of relation (3.4); the other case is proved similarly.

Since $[y, z] = 1$ in $\mathrm{A}(\{x, y, u, z\})$ (viewed as a subgraph of T_3, on the left-hand side of Fig. 27), we have the relation $[w^{-1}uw, v^{-1}xv] = 1$, as in relation (3.5). Finally, the fact that $\langle y, z\rangle = 1$ in $\mathrm{A}(\{y, v, w, z\})$, viewed as a subgraph of T_4, implies by the triangulation property that $\langle v^{-1}xv, w^{-1}uw\rangle = 1$, which proves relation (3.6). $\square$

Since A_{Y} satisfies the parabolic subgroup property, which implies the triangulation property (see the proof of Theorem 8.1), we conclude the following.

Corollary 8.6. *The functor* A_{Y} *is the largest tight functor smaller than* A, *which has the parabolic subgroup property.*

9. Geometric Braid Groups

Let T be a planar graph on $n \geq 4$ vertices. In this section we apply the ideas of Sec. 5 to define a quotient of the group $\mathrm{A}_{\mathrm{Y}}(T)$ which extends the action of the braid group on the unit disk in a manageable way. One of the main results in this section is that, although the definition makes use of the choice of a spanning sub-tree, the outcome is independent of this choice.

Let $T_0 \subseteq T$ be a spanning sub-tree. By Corollary 3.12, the subgroup $\langle T_0\rangle$ of $\mathrm{A}_{\mathrm{Y}}(T)$ is isomorphic to the braid group B_n, and so it acts on the set $\vec{\Psi}$ of directed good paths as in Sec. 6. Fix a direction for every $u \in T - T_0$, and let $\vec{\alpha}_u$ denote the corresponding element of $\vec{\Psi}$.

Fix $u \in T - T_0$. There is a unique planar path $w_1, \ldots, w_s \in T_0$ connecting the vertices of u (or rather $\vec{\alpha}_u$). Let

$$\alpha_u = \mathcal{L}(w_1 \cdots w_s) \in \langle T_0\rangle,$$

where the operator $\mathcal{L}$ was defined in (3.7). As in the case of the circle (Sec. 6), $\pi(u) = \alpha_u \in \langle T_0\rangle \subseteq \mathrm{A_Y}(T)$, and we take $x^{(u)} = u\alpha_u^{-1}$.

Definition 9.1. We set

$$G(T, T_0) = \mathcal{G}(\mathrm{A_Y}(T), \langle T_0\rangle, \vec{\Psi}, \{x^{(u)}\}_{u\in T-T_0}, \{\vec{\alpha}_u\}_{u\in T-T_0}), \tag{9.1}$$

where the operator $\mathcal{G}$ was defined in Definition 5.5.

Since $\langle T_0\rangle \cong B_n$, $G(T, T_0)$ is the largest quotient of $\mathrm{A_Y}(T)$ acting geometrically on $\vec{\Psi}$ in a way that extends the action of B_n.

Let T be a planar graph, and let T' be the graph obtained from T by a triangulation step as in Sec. 8, namely there are edges a, a' such that $T \cup \{a'\} = T' \cup \{a\}$, and a, b, a' form a minimal triangle.

Let T_0 be a spanning sub-tree of T, and let T_0' denote the spanning sub-tree of T' obtained from T_0 by the same triangulation step, namely: if $a \in T_0$ then $T_0' = T_0 - \{a\} \cup \{a'\}$; otherwise $T_0' = T_0$.

Lemma 9.2. *With $T_0 \subseteq T$ and $T_0' \subseteq T'$ as above, we have that $G(T, T_0) \cong G(T', T_0')$.*

Proof. By Theorem 8.1, we have an isomorphism $\psi: \mathrm{A_Y}(T) \cong \mathrm{A_Y}(T')$, defined by $a \mapsto b^{-1}a'b$ for a suitable $b \in T \cap T'$. The action of $\mathrm{A_Y}(T)$ and $\mathrm{A_Y}(T')$ on $\vec{\Psi}$ (through the spanning sub-trees $\{T_0\}$ and $\{T_0'\}$, respectively) commutes with ψ.

Take $H = \mathrm{A_Y}(T)$, $S = \langle T_0\rangle$ and $X = T - T_0$; and $H' = \mathrm{A_Y}(T')$, $S' = \langle T_0'\rangle$ and $Y = T' - T_0'$ in Lemma 5.8, we see that ψ induces an isomorphism $G(T, T_0) \to G(T', T_0')$ (if $a \notin T_0$ then take $\tau = b$ in the lemma; otherwise $\tau = 1$). □

Let $T^{(m)}$ denote the graph on n vertices depicted in Fig. 28, with the $n-1$ standard edges $\sigma_1, \ldots, \sigma_{n-1}$ at the bottom, and m edges, labeled $u_1, \ldots, u_m$ and numerated from bottom to top, connecting the extreme points. Let $T^{(m,0)} = \{\sigma_1, \ldots, \sigma_{n-1}\}$ denote the standard spanning sub-tree. In this case the α_u all coincide.

Lemma 9.3. *For any spanning sub-tree T_0' of $T^{(m)}$, we have that*

$$G(T^{(m)}, T_0') \cong G(T^{(m)}, T^{(m,0)}).$$

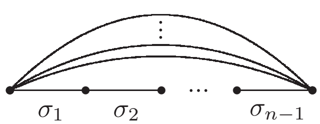

Fig. 28. The graph $T^{(m)}$.

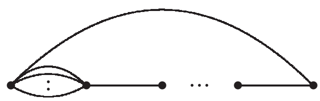

Fig. 29. The graph $\tilde{T}^{(m)}$.

Proof. For $k = 1, \dots, n-1$ and $j = 1, \dots, m$, let $S_{k,j}$ denote the spanning sub-tree of $T^{(m)}$ obtained by removing σ_k from $T^{(m,0)}$, and adding the edge u_j. Also let $S_{0,0} = T^{(m,0)}$. The $S_{k,j}$ are the only spanning sub-trees of $T^{(m)}$.

Let $\tilde{T}^{(m)}$ denote the graph obtained from $T^{(m)}$ by rotation counterclockwise, as depicted in Fig. 29, with m parallel paths replacing σ_1. Let ϕ denote the isomorphism of planar graphs from $T^{(m)}$ to $\tilde{T}^{(m)}$, and let $\tilde{S}_{k,j}$ denote the image of $S_{k,j}$ under ϕ. Clearly, $\tilde{S}_{k,j}$ are the only spanning sub-trees of $\tilde{T}^{(m)}$.

By definition, the groups $G(T^{(m)}, S_{k,j})$ are quotients of $\mathrm{A_Y}(T^{(m)})$. The action of ϕ on the graphs induces an obvious isomorphism of groups $\mathrm{A_Y}(T^{(m)}) \to \mathrm{A_Y}(\tilde{T}^{(m)})$, which carries $G(T^{(m)}, S_{k,j})$ to $G(\tilde{T}^{(m)}, \tilde{S}_{k,j})$.

For every $j = 1, \dots, m$, there is a series of triangulation steps that transforms $\tilde{T}^{(m)}$ to $T^{(m)}$, which carries $\tilde{S}_{0,0}$ to $T_{1,j}$. By Lemma 9.2, this proves $G(T^{(m)}, T^{(m,0)}) = G(T_0, S_{0,0}) \cong G(\tilde{T}^{(m)}, \tilde{S}_{0,0}) \cong G(T^{(m)}, T_{1,j})$.

Also, for every $k = 1, \dots, n-2$ and $j = 1, \dots, m$, there is a series of triangulation steps that transforms $\tilde{T}^{(m)}$ into $T^{(m)}$, carrying $\tilde{S}_{k,j}$ to $S_{k+1,j}$. This proves $G(T^{(m)}, S_{k,j}) \cong G(\tilde{T}^{(m)}, \tilde{S}_{k,j}) \cong G(T^{(m)}, S_{k+1,j})$. Together with the previous construction, we covered all possible spanning sub-trees, proving the claim. $\square$

Theorem 9.4. *Let T be a planar graph with spanning sub-tree T_0. The group $G(T, T_0)$ only depends on n and the first homology of T.*

Proof. Following the proof of Theorem 8.3, we can transform T, by a series of triangulation steps, into $T^{(m)}$ for a suitable m. In this process T_0 becomes some spanning sub-tree T_0' of $T^{(m)}$. By Lemma 9.2, $G(T, T_0) \cong G(T^{(m)}, T_0')$. By Lemma 9.3, $G(T^{(m)}, T_0') \cong G(T^{(m)}, T^{(m,0)})$, which only depends on n and m. $\square$

Corollary 9.5. *The group $G(T, T_0)$ is independent of T_0.*

10. The General Case

Let $n \geq 5$. Our aim in this section is to compute $G(T, T_0)$, defined in Definition 9.1, for an arbitrary planar graph T and spanning sub-tree T_0. In light of Theorem 9.4, we may assume T is the graph $T^{(m)}$ of Fig. 28, for a suitable $m \geq 0$, with spanning sub-tree $T_0 = T^{(m,0)} = \{\sigma_1, \dots, \sigma_{n-1}\}$. In this case the α_u of the definition all coincide with the element α given in Eq. (6.1), and we can apply the results of Sec. 6 more easily.

10.1. *A presentation of G*

In order to compute $G = G(T^{(m)}, T_0)$, we need to establish a presentation. As noted in Remark 3.4,

$$\langle T_0 \rangle \cong B_n. \tag{10.1}$$

By definition, the other defining relations of $\mathrm{A}(T^{(m)})$ are, for every $j = 1, \dots, m$ and $1 < \ell < n-1$,

$$\langle \sigma_1, u_j \rangle = 1, \tag{10.2}$$

$$\langle \sigma_{n-1}, u_j \rangle = 1, \tag{10.3}$$

$$[\sigma_\ell, u_j] = 1. \tag{10.4}$$

To obtain a presentation of $\mathrm{A}_{\mathrm{Y}}(T^{(m)})$, we must add, for every $1 \le i < j \le m$,

$$\langle \sigma_1 u_i \sigma_1^{-1}, u_j \rangle = 1, \tag{10.5}$$

$$\langle \sigma_{n-1} u_j \sigma_{n-1}^{-1}, u_i \rangle = 1, \tag{10.6}$$

$$[\sigma_1 u_i \sigma_1^{-1}, \sigma_{n-1} u_j \sigma_{n-1}^{-1}] = 1. \tag{10.7}$$

Write $u_i = x^{(i)}\alpha$, defining new elements $x^{(i)}$, $i = 1, \dots, m$. By definition G is the geometric quotient of $\mathrm{A}_{\mathrm{Y}}(T^{(m)})$ with respect to the action on the disk, and so we add the relations

$$[x^{(j)}, \sigma_\ell] = 1, \tag{10.8}$$

$$[x^{(j)}, \sigma_{n-1}\sigma_1^{-1}\alpha\sigma_1\sigma_{n-1}^{-1}] = 1, \tag{10.9}$$

$$[x^{(j)}, \alpha^2] = 1, \tag{10.10}$$

for every $j = 1, \dots, m$ and $1 < \ell < n-1$ (see Remark 4.5).

Relations (10.2)–(10.10), together with (10.1), provide a presentation of G. Relations (10.2)–(10.4) were treated in Sec. 6; they allow us to define $x_\omega^{(j)}$ as in Eq. (5.1). The relations satisfied by $\{x_\omega^{(j)} : \omega \in \vec{\Psi}\}$, for fixed j, are summarized in Theorem 6.11. The remaining difficulty is in the interaction of the $x_\omega^{(j)}$ for distinct values of j. In order to simplify relations (10.5)–(10.7), we substitute $u_i = x^{(i)}\alpha$, and obtain:

$$\langle \sigma_1 x_\alpha^{(i)} \alpha \sigma_1^{-1}, x_\alpha^{(j)} \alpha \rangle = 1,$$
$$\langle \sigma_{n-1} x_\alpha^{(j)} \alpha \sigma_{n-1}^{-1}, x_\alpha^{(i)} \alpha \rangle = 1,$$
$$[\sigma_1 x_\alpha^{(i)} \alpha \sigma_1^{-1}, \sigma_{n-1} x_\alpha^{(j)} \alpha \sigma_{n-1}^{-1}] = 1,$$

for $1 \le i < j \le m$ and $1 < \ell < n-1$.

To simplify these further, recall the action of B_n on the $x_\omega^{(j)}$ by conjugation, described in Eq. (5.2). We rewrite the relations in these terms, applying Remark 5.3:

$$x^{(i)}_{\sigma_1^{-1}(\vec{\alpha})} \cdot x^{(j)}_{(\sigma_1\alpha^{-1}\sigma_1^{-1})(\vec{\alpha})} \cdot x^{(i)}_{(\sigma_1^{-1}\alpha^{-1}\sigma_1\alpha^{-1}\sigma_1^{-1})(\vec{\alpha})} = x^{(j)}_{\vec{\alpha}} \cdot x^{(i)}_{(\sigma_1^{-1}\alpha^{-1})(\vec{\alpha})} \cdot x^{(j)}_{(\alpha^{-1}\sigma_1^{-1})(\vec{\alpha})},$$

$$x^{(j)}_{\sigma_{n-1}^{-1}(\vec{\alpha})} \cdot x^{(i)}_{(\sigma_{n-1}\alpha^{-1}\sigma_{n-1}^{-1})(\vec{\alpha})} \cdot x^{(j)}_{(\alpha\sigma_{n-1}^{-1}\alpha^{-2}\sigma_{n-1}^{-1})(\vec{\alpha})} = x^{(i)}_{\vec{\alpha}} \cdot x^{(j)}_{(\sigma_{n-1}^{-1}\alpha^{-1})(\vec{\alpha})} \cdot x^{(i)}_{(\alpha^{-1}\sigma_{n-1}^{-1})(\vec{\alpha})},$$

$$x^{(i)}_{\sigma_1^{-1}(\vec{\alpha})} \cdot x^{(j)}_{(\sigma_{n-1}^{-1}\sigma_1\alpha^{-1}\sigma_1^{-1})(\vec{\alpha})} = x^{(j)}_{\sigma_{n-1}^{-1}(\vec{\alpha})} \cdot x^{(i)}_{(\sigma_1^{-1}\sigma_{n-1}\alpha^{-1}\sigma_{n-1}^{-1})(\vec{\alpha})}.$$

Acting with $\sigma_1\alpha\sigma_1^{-1}\sigma_{n-1}\dots\sigma_3$ on the first equation, we obtain the relation

$$x^{(i)}_{\overleftarrow{\sigma}_2} \cdot x^{(j)}_{\overrightarrow{\omega}} \cdot x^{(i)}_{\overleftarrow{\sigma}_1} = x^{(j)}_{\overrightarrow{\sigma}_1} \cdot x^{(i)}_{\overleftarrow{\omega'}} \cdot x^{(j)}_{\overrightarrow{\sigma}_2}, \tag{10.11}$$

where $\overrightarrow{\omega}$ and $\overleftarrow{\omega'}$, as well as the other directed paths, are depicted in Fig. 30. Similarly, acting on the second equation with $\sigma_{n-1}\dots\sigma_2\sigma_{n-1}\dots\sigma_3\sigma_1^{-1}$, we obtain the relation

$$x^{(j)}_{\overrightarrow{\sigma}_2} \cdot x^{(i)}_{\overleftarrow{\omega}} \cdot x^{(j)}_{\overrightarrow{\sigma}_1} = x^{(i)}_{\overleftarrow{\sigma}_1} \cdot x^{(j)}_{\overrightarrow{\omega'}} \cdot x^{(i)}_{\overleftarrow{\sigma}_2}. \tag{10.12}$$

Finally, since $(\sigma_{n-1}^{-1}\sigma_1\alpha^{-1}\sigma_1^{-1}\sigma_{n-1})(\vec{\alpha}) = \vec{\alpha}$, the third equation is equivalent to

$$[x^{(i)}_{\overrightarrow{\sigma}_1}, x^{(j)}_{\overrightarrow{\sigma}_3}] = 1, \tag{10.13}$$

namely (by the transitive action on $\vec{\Psi}$), $x^{(i)}_\rho$ and $x^{(j)}_{\rho'}$ commute whenever they are based on disjoint paths ρ and ρ'.

Proposition 10.1. *If $n \geq 5$, then for each i, $z_i = [x^{(i)}_{\overrightarrow{\sigma}_1}, x^{(i)}_{\overrightarrow{\sigma}_2}]$ is central in G.*

Proof. By Proposition 6.8, $[x^{(i)}_\omega, x^{(i)}_{\omega'}]$ is the same element of the group, whenever ω and ω' form a partial frame. This implies z_i that commutes with B_n. Now, for every generator $x^{(j)}_{\omega''}$, we may choose the partial frame to be disjoint from ω'' (taking

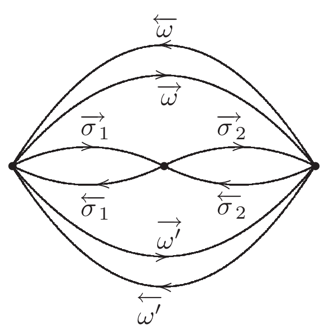

Fig. 30. Notation for the basic relations.

the three endpoints of the partial frame differing from the two endpoints of ω'', as $n \geq 5$), so we are done by Eq. (10.13). □

Applying the basic relations from Sec. 6, in particular Corollary 6.6, as well as Proposition 10.1, relations (10.11) and (10.12) become

$$x_{\vec{\sigma_2}}^{-(i)} \cdot x_{\vec{\sigma_1}}^{(j)} \cdot x_{\vec{\sigma_2}}^{(j)} \cdot x_{\vec{\sigma_1}}^{-(i)} = x_{\vec{\sigma_1}}^{(j)} \cdot x_{\vec{\sigma_2}}^{-(i)} \cdot x_{\vec{\sigma_1}}^{-(i)} \cdot x_{\vec{\sigma_2}}^{(j)}, \tag{10.14}$$

$$x_{\vec{\sigma_2}}^{(j)} \cdot x_{\vec{\sigma_1}}^{-(i)} \cdot x_{\vec{\sigma_2}}^{-(i)} \cdot x_{\vec{\sigma_1}}^{(j)} = x_{\vec{\sigma_1}}^{-(i)} \cdot x_{\vec{\sigma_2}}^{(j)} \cdot x_{\vec{\sigma_1}}^{(j)} \cdot x_{\vec{\sigma_2}}^{-(i)}, \tag{10.15}$$

where $x_{\vec{\sigma_1}}^{-(i)}$ is a shorthand for ${x_{\vec{\sigma_1}}^{(i)}}^{-1}$. These relations can be brought to the form

$$[x_{\vec{\sigma_2}}^{(j)}, x_{\vec{\sigma_1}}^{-(i)}] = [x_{\vec{\sigma_1}}^{-(j)}, x_{\vec{\sigma_2}}^{(i)}], \tag{10.16}$$

$$[x_{\vec{\sigma_2}}^{-(i)}, x_{\vec{\sigma_1}}^{(j)}] = [x_{\vec{\sigma_1}}^{(i)}, x_{\vec{\sigma_2}}^{-(j)}]. \tag{10.17}$$

10.2. *The kernel of $G \to B_n$: A first presentation*

Let

$$K = \mathcal{K}(\mathrm{A_Y}(T), B_n, \vec{\Psi}, \{x_u\}, \{\vec{\alpha}_u\}) \tag{10.18}$$

be the group defined in Definition 5.5, namely $K = \mathrm{Ker}(G \to B_n)$. The computation done so far can be summarized as follows.

Proposition 10.2. *The group K has the following presentation. The generators are $x_\omega^{(i)}$ for $i = 1, \ldots, m$ and $\omega \in \vec{\Psi}$; and z_i for $i = 1, \ldots, m$.*

The relations are:

(i) *z_i are central and ${z_i}^2 = 1$;*
(ii) *if ω_1 and ω_2 form a partial frame, then $[x_{\vec{\omega_1}}^{(i)}, x_{\vec{\omega_2}}^{(i)}] = z_i$;*
(iii) *if ω_1 and ω_2 are disjoint, then $[x_{\vec{\omega_1}}^{(i)}, x_{\vec{\omega_2}}^{(j)}] = 1$ for every i, j; and*
(iv) *if ω_1 and ω_2 intersect in one vertex, then*

$$[x_{\vec{\omega_1}}^{(i)}, x_{\vec{\omega_2}}^{-(j)}] = [x_{\vec{\omega_2}}^{-(i)}, x_{\vec{\omega_1}}^{(j)}]$$

for every i, j.

Proof. Relations (i) and (ii) come from Sec. 6. Relation (iii) was obtained at (10.13). If $i = j$, then relation (iv) follows from (ii); moreover, taking the inverse in relation (iv) switches the roles of i and j, and so we may assume $i < j$. There are two cases to consider: if the head of ω_1 touches the tail of ω_2, then we are done by taking $\omega_1 = \sigma_1$ and $\omega_2 = \sigma_2$ in (10.17); if the head of ω_2 touches the tail of ω_1, we are done by taking $\omega_1 = \sigma_2$ and $\omega_2 = \sigma_1$ in (10.16). □

10.3. *A finite presentation for the kernel*

The presentation of the previous subsection, which has a geometric flavor, has infinitely many generators and infinitely many relations. In Corollary 6.7 we saw that every $x_{\vec{\omega}}^{(i)}$ can be expressed in terms of the $x_r^{(i)} = x_{\vec{\sigma}_r}^{(i)}$, $r = 1, \dots, n-1$, so that K is finitely generated. We now show that K is in fact finitely presented.

Proposition 10.3. *The group K defined in* (10.18) *has the following presentation. Generators*: $x_r^{(i)}$ *for* $i = 1, \dots, m$, $r = 1, \dots, n-1$, *and* z_i *for* $i = 1, \dots, m$.

Relations: *for* $i, j = 1, \dots, m$ *and* $r, s = 1, \dots, n$,

(i) z_i *are central and* ${z_i}^2 = 1$;
(ii) $[x_r^{(i)}, x_{r+1}^{(i)}] = z_i$ *for* $1 \le r < n-1$;
(iii) $[x_r^{(i)}, x_s^{(j)}] = 1$ *if* $|r-s| > 1$;
(iv) $[x_r^{(i)} x_{r-1}^{(i)}, x_r^{(j)} x_{r+1}^{(j)}] = 1$ *for* $1 < r < n-1$; *and*
(v) $[x_r^{(i)}, x_{r+1}^{-(j)}] = [x_{r+1}^{-(i)}, x_r^{(j)}]$ *for* $1 \le r < n-1$.

Proof. The presentation claimed here can be compared to that of Proposition 10.2 by identifying $x_r^{(i)}$ of the current one with $x_{\vec{\sigma}_r}^{(i)}$ of the previous one. Clearly, every relation in the current presentation is assumed to hold for the $x_{\vec{\sigma}_r}^{(i)}$ (relations (i), (ii), (iii) and (v) follow from 10.2(i)–10.2(iv), and (iv) follows from 10.2(iii) by taking $\omega_1 = \sigma_r(\vec{\sigma}_{r-1})$ and $\omega_2 = \sigma_r(\vec{\sigma}_{r+1})$). On the other hand, every class of relations in Proposition 10.2 has a representative in the current presentation. The presentation of Proposition 10.2 is invariant under the action of B_n, being phrased in terms of paths.

Therefore, it is enough to show that the current presentation is invariant under the action of B_n, induced from the identification $x_r^{(i)} = x_{\vec{\sigma}_r}^{(i)}$. Similarly to Definition 6.10, the action is defined, for every i, by $\sigma_t(z_i) = z_i$ and

$$\sigma_t(x_r^{(i)}) = \begin{cases} x_t^{(i)} x_{t-1}^{(i)}, & r = t-1, \\ z_i x_t^{-(i)}, & r = t, \\ x_t^{(i)} x_{t+1}^{(i)}, & r = t+1, \\ x_r^{(i)}, & |r-t| > 1. \end{cases} \tag{10.19}$$

One can easily check that this action, which reduces to the standard identification of B_n as a subgroup of the automorphism group of the free group (if all z_i are sent to 1), is, for each i, a well defined action of B_n on the free group generated by the $x_r^{(i)}$ and z_i, modulo z_i being central.

The action respects (i) and (ii) — this is easy to check. The only difficulty in case (iii) is for $r = t-1$ and $s = t+1$, where we have

$$[\sigma_t(x_{t-1}^{(i)}), \sigma_t(x_{t+1}^{(j)})] = [x_t^{(i)} x_{t-1}^{(i)}, x_t^{(j)} x_{t+1}^{(j)}] = 1$$

by relation (iv). Similarly,

$$[\sigma_t^{-1}(x_{t-1}^{(i)}), \sigma_t^{-1}(x_{t+1}^{(j)})] = [x_{t-1}^{(i)}x_t^{(i)}, x_{t+1}^{(j)}x_t^{(j)}],$$

which equals $[x_t^{(i)}x_{t-1}^{(i)}, x_t^{(j)}x_{t+1}^{(j)}] = 1$ by relations (ii) and (i).

Next, we need to show that relation (iv) is preserved under the σ_t (and σ_t^{-1}, which can be resolved in the same manner). If $t \leq r-3$ or $t \geq r+3$, the action is trivial and there is nothing to prove. For $t = r-2$ we obtain

$$[\sigma_{r-2}(x_r^{(i)})\sigma_{r-2}(x_{r-1}^{(i)}), \sigma_{r-2}(x_r^{(j)})\sigma_{r-2}(x_{r+1}^{(j)})] = [x_r^{(i)}x_{r-1}^{(i)}x_{r-2}^{(i)}, x_r^{(j)}x_{r+1}^{(j)}],$$

where $x_r^{(i)}x_{r-1}^{(i)}x_{r-2}^{(i)} = (x_r^{(i)}x_{r-1}^{(i)}) \cdot x_{r-2}^{(i)}$ commutes with $x_r^{(j)}x_{r+1}^{(j)}$ by relations (iv) and (iii). The case $t = r+2$ is dealt with in a similar manner.

For $t = r-1$ we obtain

$$\begin{aligned}
&[\sigma_{r-1}(x_r^{(i)})\sigma_{r-1}(x_{r-1}^{(i)}), \sigma_{r-1}(x_r^{(j)})\sigma_{r-1}(x_{r+1}^{(j)})]^{-1} \\
&\quad = [z_x x_r^{(i)}, x_{r-1}^{(j)}x_r^{(j)}x_{r+1}^{(j)}]^{-1} \\
&\quad = [x_{r-1}^{(j)}x_r^{(j)}x_{r+1}^{(j)}, x_r^{(i)}] \\
&\quad = x_{r-1}^{(j)}x_r^{(j)}x_{r+1}^{(j)}x_r^{(i)}x_{r+1}^{-(j)}x_r^{-(j)}x_{r-1}^{-(j)}x_r^{-(i)} \\
&\quad = x_{r-1}^{(j)}x_r^{(j)}x_r^{(i)}[x_r^{-(i)}, x_{r+1}^{(j)}]x_r^{-(j)}x_{r+1}^{-(j)}x_r^{-(i)} \\
&\quad \overset{\text{(v)}}{=} x_{r-1}^{(j)}x_r^{(j)}x_r^{(i)}x_{r+1}^{(i)}x_{r+1}^{-(i)}[x_{r+1}^{(i)}, x_r^{-(j)}]x_r^{-(j)}x_{r+1}^{-(j)}x_r^{-(i)} \\
&\quad \overset{\text{(iv)}}{=} x_r^{(i)}x_{r+1}^{(i)}x_{r-1}^{(j)}x_r^{(j)}x_{r+1}^{-(i)}[x_{r+1}^{(i)}, x_r^{-(j)}]x_r^{-(j)}x_{r+1}^{-(j)}x_r^{-(i)} \\
&\quad = x_r^{(i)}[x_{r+1}^{(i)}, x_{r-1}^{(j)}]x_r^{-(i)} \\
&\quad \overset{\text{(iii)}}{=} 1,
\end{aligned}$$

where the proof for $t = r+1$ is similar. For later use, we record the identity

$$[x_{r-1}^{(j)}x_r^{(j)}x_{r+1}^{(j)}, x_r^{(i)}] = 1 \tag{10.20}$$

which was proved as part of the computation above. Finally, applying σ_r to relation (iv) transfers $x_r^{(i)}x_{r-1}^{(i)}$ and $x_r^{(j)}x_{r+1}^{(j)}$ to $z_i x_{r-1}^{(i)}$ and $z_j x_{r+1}^{(j)}$, which clearly commute.

It remains to act on relation (v). Clearly, the action of σ_t is trivial if $t < r-1$ or $t > r+2$. If $t = r-1$, then $x_r^{(i)}$ is mapped to $x_{r-1}^{(i)}x_r^{(i)}$ while $x_{r+1}^{(j)}$ is fixed; but $x_{r-1}^{(i)}$ commutes with $x_{r+1}^{(j)}$ by relation (iii), so the commutator relation holds. The same proof applies for $t = r+2$. For $t = r+1$ we have

$$\begin{aligned}
&[\sigma_{r+1}(x_r^{(i)}), \sigma_{r+1}(x_{r+1}^{-(j)})][\sigma_{r+1}(x_{r+1}^{-(i)}), \sigma_{r+1}(x_r^{(j)})]^{-1} \\
&\quad = [x_{r+1}^{(i)}x_r^{(i)}, z_j x_{r+1}^{(j)}][z_i x_{r+1}^{(i)}, x_{r+1}^{(j)}x_r^{(j)}]^{-1}
\end{aligned}$$

$$= [x_{r+1}^{(i)} x_r^{(i)}, x_{r+1}^{(j)}][x_{r+1}^{(i)}, x_{r+1}^{(j)} x_r^{(j)}]^{-1}$$
$$= x_{r+1}^{(i)} x_r^{(i)} x_{r+1}^{(j)} x_r^{-(i)} x_{r+1}^{-(i)} x_r^{(j)} x_{r+1}^{(i)} x_r^{-(j)} x_{r+1}^{-(j)} x_{r+1}^{-(i)}$$
$$= x_{r+1}^{(i)} x_{r+1}^{(j)} [x_{r+1}^{-(j)}, x_r^{(i)}][x_r^{(j)}, x_{r+1}^{-(i)}]^{-1} x_{r+1}^{-(j)} x_{r+1}^{-(i)} = 1$$

by relation (v). A similar computation handles the case $t = r$, and we are done. □

10.4. *A second presentation for the kernel*

In order to simplify the presentation of K, we make the following substitution. For $i = 1, \dots, m$ and $r = 1, \dots, n-1$, set

$$a_r^{(i)} = x_{n-1}^{(i)} \cdots x_r^{(i)};$$

to simplify the notation, we also set $a_n^{(i)} = 1$. It easily follows that

$$x_r^{(i)} = a_{r+1}^{-(i)} a_r^{(i)}, \tag{10.21}$$

so that K is generated by the $a_r^{(i)}$.

Proposition 10.4. *The group K has the following presentation. Generators: $a_r^{(i)}$ for $i = 1, \dots, m$, $r = 1, \dots, n$, and z_i for $i = 1, \dots, m$.*

Relations: $a_n^{(i)} = 1$ for every $i = 1, \dots, m$. Furthermore, for $i, j = 1, \dots, m$,

(i) *z_i are central and ${z_i}^2 = 1$;*
(ii) *$[a_{r+2}^{(i)}, a_r^{(i)}][a_r^{(i)}, a_{r+1}^{(i)}][a_{r+1}^{(i)}, a_{r+2}^{(i)}] = z_i$ for $1 \le r \le n-2$;*
(iii) *$[a_{s+1}^{(j)}, a_r^{(i)}][a_r^{(i)}, a_s^{(j)}][a_s^{(j)}, a_{r+1}^{(i)}][a_{r+1}^{(i)}, a_{s+1}^{(j)}] = 1$ if $1 \le r, s \le n-1$ and $|r - s| > 1$;*
(iv) *$[a_{r+2}^{(j)}, a_{r-1}^{(i)}][a_{r-1}^{(i)}, a_r^{(j)}][a_r^{(j)}, a_{r+1}^{(i)}][a_{r+1}^{(i)}, a_{r+2}^{(j)}] = 1$ for $2 \le r \le n-2$; and*
(v) *for $1 \le r \le n-2$, we have*

$$[a_{r+1}^{(j)}, a_r^{(i)}][a_r^{(i)}, a_{r+2}^{(j)}][a_{r+2}^{(j)}, a_{r+1}^{(i)}] = [a_{r+1}^{(j)}, a_{r+2}^{(i)}][a_{r+2}^{(i)}, a_r^{(j)}][a_r^{(j)}, a_{r+1}^{(i)}].$$

Proof. This follows by substitution in the previous set of relations, where we used the identity

$$[\alpha^{-1}\beta, \gamma^{-1}\delta] = \alpha^{-1}\gamma^{-1}[\gamma, \beta][\beta, \delta][\delta, \alpha][\alpha, \gamma]\gamma\alpha. \tag{10.22}$$

Note that in relation (iv) we substituted $x_{r+1}^{(j)} x_r^{(j)}$ rather than $x_r^{(j)} x_{r+1}^{(j)}$; but $x_r^{(j)}, x_{r+1}^{(j)}$ commute. □

Proposition 10.5. *Fix $1 \le i, j \le m$. The elements $[a_r^{(i)}, a_s^{(j)}]$, for $1 \le r, s < n$, $r \ne s$, are all equal.*

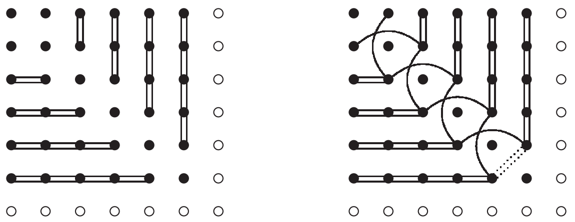

Fig. 31. An illustration of the proof of Proposition 10.5, for $n = 7$. A black bullet symbol in the (r, s) coordinate stands for $\gamma_{rs}^{(ij)}$; white bullets represent $\gamma_{rs}^{(ij)} = 1$.

Proof. Write $\gamma_{r,s}^{(ij)} = [a_r^{(i)}, a_s^{(j)}]$. Relation 10.4(iii) translates to the equality $\gamma_{r+1,s}^{-(ij)}\gamma_{r+1,s+1}^{(ij)} = \gamma_{r,s}^{-(ij)}\gamma_{r,s+1}^{(ij)}$ whenever $|r - s| > 1$. Since $\gamma_{n,s}^{(ij)} = 1$ for every s, reverse induction shows that $\gamma_{r,s+1}^{(ij)} = \gamma_{r,s}^{(ij)}$ for every $r > s + 1$. In a similar way, by first taking $s = n$, we have

$$\gamma_{r+1,s}^{(ij)} = \gamma_{r,s}^{(ij)} \quad \text{for} \ \ 2 \leq r + 1 < s \leq n. \tag{10.23}$$

These equalities are illustrated by connecting bullets representing equal elements, at the left-hand side of Fig. 31.

Relation 10.4(iv) translates to

$$\gamma_{r-1,r+2}^{-(ij)}\gamma_{r-1,r}^{(ij)}\gamma_{r+1,r}^{-(ij)}\gamma_{r+1,r+2}^{(ij)} = 1 \quad \text{for } 1 < r < n - 1. \tag{10.24}$$

Relation (10.23) implies that $\gamma_{r-1,r+2}^{(ij)} = \gamma_{r,r+2}^{(ij)} = \gamma_{r+1,r+2}^{(ij)}$, and so (10.24) gives $\gamma_{r-1,r}^{(ij)} = \gamma_{r+1,r}^{(ij)}$ for $1 < r < n - 1$. Switching i and j in relation 10.4(iv), we obtain in a symmetric way $\gamma_{r,r-1}^{(ij)} = \gamma_{r,r+1}^{(ij)}$ for $1 < r < n - 1$. In this manner we obtain the curved connections shown in the right-hand side of Fig. 31, again connecting bullets that represent equal elements.

Finally, put $r = n - 2$ in relation 10.4(v); noting that $a_n^{(i)} = a_n^{(j)} = 1$, we obtain

$$[a_{n-1}^{(j)}, a_{n-2}^{(i)}] = [a_{n-2}^{(j)}, a_{n-1}^{(i)}],$$

namely, $\gamma_{n-2,n-1}^{(ij)} = \gamma_{n-1,n-2}^{(ij)}$, which is the double-dotted line in the diagram. The (non-empty) off-diagonal bullets in Fig. 31 are now in one connected component, proving the statement. $\square$

Corollary 10.6. *For every* $1 \leq i \leq m$ *and* $1 \leq r, s < n$, $r \neq s$, *we have*

$$[a_r^{(i)}, a_s^{(i)}] = z_i. \tag{10.25}$$

Proof. By the proposition, $[a_r^{(i)}, a_s^{(i)}] = \gamma^{(ii)}$ is independent of r, s, as long as $r \neq s$. Switching r and s, it follows that $\gamma^{(ii)} = \gamma^{-(ii)}$.

Taking $r = 1$ in relation 10.4(ii) results in the relation

$$[a_3^{(i)}, a_1^{(i)}][a_1^{(i)}, a_2^{(i)}][a_2^{(i)}, a_3^{(i)}] = z_i.$$

By Proposition 10.5 (which does apply when $i = j$), this implies $(\gamma^{(ii)})^3 = z_i$, so $\gamma^{(ii)} = z_i$ by the first remark. □

10.5. *The structure of K*

The results of Proposition 10.5 and Corollary 10.6 can be summarized as follows.

Corollary 10.7. *The group K is generated by $a_r^{(i)}$ for $i = 1, \dots, m$, $r = 1, \dots, n-1$, subject to the relations*

(i) $[a_r^{(i)}, a_s^{(j)}] = [a_{r'}^{(i)}, a_{s'}^{(j)}]$ *for any $r \neq s$ and $r' \neq s'$ and any $i, j = 1, \dots, m$.*
(ii) $z_i = [a_r^{(i)}, a_s^{(i)}]$ *(independent of $r \neq s$) is central and has square equal to* 1, *for any i.*

Corollary 10.8. *If A is the normal subgroup of K generated by $[a_r^{(i)}, a_s^{(j)}]$ for $r \neq s$, then $K/A \cong \langle a_1^{(1)}, \dots, a_1^{(m)} \rangle \times \cdots \times \langle a_{n-1}^{(1)}, \dots, a_{n-1}^{(m)} \rangle$, which is a direct product of $n-1$ copies of the free group $\mathbb{F}_m$.*

Recall from Sec. 2 that $F_{m,n}$ is defined as a certain subgroup of $\mathbb{F}_m^n$, where $\mathbb{F}_m$ denotes the free group on m generators; also recall the group $A_{m,n} \cong F_{m,n}$ defined there.

Theorem 10.9. *The group K is a central extension of $F_{m,n}$ by $(\mathbb{Z}/2\mathbb{Z})^m$.*

Proof. Let A_0 denote the subgroup of K generated by the commutators $A_0 = \langle [a_r^{(i)}, a_s^{(i)}] \rangle$ ($i = 1, \dots, m$, $r \neq s$). Clearly A_0 is a central subgroup of exponent 2 and rank at most m.

Let $a_1, \dots, a_m$ denote the generators of $\mathbb{F}_m$. We define a map $K \to \mathbb{F}_m^n$ by

$$a_r^{(i)} \mapsto (1, \dots, 1, a_i, 1, \dots, 1, a_i^{-1}) \tag{10.26}$$

(non-trivial entries in the rth and nth places). This clearly maps K onto $F_{m,n}$. For any $r \neq s$, $1 \leq r, s < n$, the commutator $[a_r^{(i)}, a_s^{(j)}]$ maps to $(1, \dots, 1, [a_i^{-1}, a_j^{-1}])$, which is independent of r and s. In particular $[a_r^{(i)}, a_s^{(i)}]$ maps to the identity elements, and so the induced epimorphism $K/A_0 \to F_{m,n}$ is well defined. This could also be deduced from Proposition 2.2 by constructing appropriate maps from K to $\mathbb{F}_m^{n-1}$ and to $\mathbb{F}_m$.

To show that this is an isomorphism, we define a map $A_{m,n} \to K/A_0$ by $x_{rs}^{(i)} \mapsto a_s^{-(i)} a_r^{(i)}$. To see that this is well defined, we need to verify the defining equations in K/A_0: Eq. (2.2) is trivial. Equation (2.3) translates to

$$a_s^{-(i)} a_r^{(i)} a_t^{-(i)} a_s^{(i)} = a_t^{-(i)} a_r^{(i)},$$

which follows from the fact that $[a_s^{(i)}, a_r^{(i)}]$, $[a_s^{(i)}, a_t^{(i)}]$ and $[a_t^{(i)}, a_r^{(i)}]$ are central, and have trivial product (in fact in K the product is z_i); Eq. (2.4) is checked similarly.

By Proposition 10.5, if $u, t \neq r, s$, then

$$\begin{aligned}&[a_u^{(j)}, a_r^{(i)}][a_r^{(i)}, a_t^{(j)}][a_t^{(j)}, a_s^{(i)}][a_s^{(i)}, a_u^{(j)}]\\ &\quad = [a_t^{(j)}, a_r^{(i)}][a_r^{(i)}, a_t^{(j)}][a_t^{(j)}, a_s^{(i)}][a_s^{(i)}, a_t^{(j)}] = 1,\end{aligned}$$

this proves Eq. (2.5), which is

$$[a_s^{-(i)} a_r^{(i)}, a_u^{-(j)} a_t^{(j)}] = 1,$$

by Eq. (10.22).

It remains to show that $\operatorname{rank}(A_0) = m$. Let $R = k \oplus V_1 \oplus V_2$ where k is a field of characteristic 2, V_1 is the k-vector space spanned by the $m(n-1)$ variables $\alpha_r^{(i)}$, and V_2 is the k-vector space spanned by m variables $\gamma_i (i = 1, \dots, m)$. Make R into an associative, non-commutative k-algebra by asserting that $\alpha_r^{(i)} \alpha_s^{(j)}$ equals γ_i if $j = i$ and $r < s$, and zero otherwise; and that $V_1 V_2 = V_2 V_1 = V_2 V_2 = 0$.

Now define a map $\phi\colon K \to R^\times$ by $a_r^{(i)} \mapsto 1 + \alpha_r^{(i)}$. It is a standard and easy fact that $[a_r^{(i)}, a_s^{(j)}] \mapsto 1 + \alpha_r^{(i)} \alpha_s^{(j)} + \alpha_r^{(i)} \alpha_s^{(j)}$. For $i \neq j$ we have $[a_r^{(i)}, a_s^{(j)}] \mapsto 1$, while $[a_r^{(i)}, a_s^{(i)}] \mapsto 1 + \gamma_i$ whenever $r \neq s$; therefore, the map is well defined. Finally, the subgroup A_0 is mapped onto $1 + V_2$, which is clearly of rank m. $\square$

Let R^1 be the multiplicative subgroup $1 + V_1 + V_2$ of the ring R defined in the proof above. Note that R^1 is a central extension of $1 + V_1 \cong (\mathbb{Z}/2\mathbb{Z})^{m(n-1)}$ by $1 + V_2 \cong (\mathbb{Z}/2\mathbb{Z})^m$.

Corollary 10.10. *K is a pull-back of the diagram*

$$\begin{tikzcd} ? \arrow[r, dotted] \arrow[d, dotted] & F_{m,n} \arrow[d] \\ R^1 \arrow[r] & (\mathbb{Z}/2\mathbb{Z})^m. \end{tikzcd}$$

In particular, the word problem is solvable in K.

10.6. *Summary: the structure of G*

By definition, the group $G = G(T^{(m)}, T^{(m,0)})$ defined in (9.1) is a semidirect product

$$G = B_n \ltimes K_{m,n},$$

where the action of $B_n = \{\sigma_1, \dots, \sigma_{n-1}\} \subseteq G$ on

$$K_{m,n} = \langle a_r^{(i)}, z_i : i = 1, \dots, m;\ r = 1, \dots, n \rangle$$

is given in (10.19), noting that $x_r^{(i)}$ are defined in (10.21), and a presentation for $K_{m,n}$ is given in Corollary 10.7. Combining the short exact sequence

$$1 \to K_{m,n} \to G \to B_n \to 1$$

with the short exact sequence (2.6), we obtain the commutative diagram given in Fig. 32. The right-hand column is the standard cover $B_n \to S_n$, whose kernel P_n is

$$\begin{tikzcd}
(\mathbb{Z}/2\mathbb{Z})^m \arrow[r, hook] \arrow[d, hook] & P_n^m \arrow[r] \arrow[d, hook] & P_n \arrow[d, hook] \\
K_{m,n} \arrow[r, hook] \arrow[d] & G \arrow[r] \arrow[d] & B_n \arrow[d] \\
F_{m,n} \arrow[r, hook] & \mathrm{C}_Y(T) \arrow[r] & S_n
\end{tikzcd}$$

Fig. 32.

the group of pure braids. In the left-hand column, $K_{m,n}$ is a central extension, where the epimorphism $K_{m,n} \to F_{m,n}$ was defined in (10.26), and the monomorphism of $(\mathbb{Z}/2\mathbb{Z})^m$ into $K_{m,n}$ is to the subgroup $\langle z_1, \ldots, z_m \rangle \subseteq K_{m,n}$, for z_i of (10.25). The map $\mathrm{A}_{\mathrm{Y}}(T) \to \mathrm{C}_{\mathrm{Y}}(T)$, appearing in the middle column of Fig. 5, induces an epimorphism from G to $\mathrm{C}_{\mathrm{Y}}(T)$, whose kernel is denoted here by P_n^m. By Proposition 10.1, P_n^m is a central extension of P_n by $(\mathbb{Z}/2\mathbb{Z})^m$.

From the short exact sequence (2.1) and the fact that the word problem is solvable in B_n, we obtain the following corollary.

Corollary 10.11. *The word problem is solvable in G.*

References

[1] M. Amram, D. Goldberg, M. Teicher and U. Vishne, The fundamental group of a Galois cover of $\mathbb{CP}^1 \times \mathbb{T}$, *Algebr. Geom. Topol.* **2** (2002) 403–432.

[2] M. Amram, M. Teicher and U. Vishne, The Coxeter quotient of the fundamental group of a Galois cover of $\mathbb{T} \times \mathbb{T}$, *Comm. Algebra* **34** (2006) 89–106.

[3] M. Amram, M. Teicher and U. Vishne, The fundamental group of Galois cover of the surface $\mathbb{T} \times \mathbb{T}$, *Internat. J. Algebra Comput.* **18**(8) (2008) 1259–1282.

[4] J. Birman, *Braids, Links and Mapping Class Groups* (Princeton University Press, 1975).

[5] J. Crisp, Automorphisms and abstract commensurators of 2-dimensional Artin groups, *Geom. Topol.* **9**(32) (2005) 1381–1441.

[6] J. Crisp and L. Paris, Artin groups of type B and D, *Adv. Geom.* **5**(4) (2005) 607–636.

[7] G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, *J. London Math. Soc.* **24** (1949) 247–254.

[8] C. Liedtke, *On Fundamental Groups of Galois Closures of Generic Projections*, Bonner Mathematische Schriften [Bonn Mathematical Publications], Vol. 367, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn (2004).

[9] M. Lönne, Fundamental group of discriminant complements of Brieskorn–Pham polynomials, *C. R. Math. Acad. Sci. Paris* **345**(2) (2007) 93–96.

[10] B. Moishezon, A. Robb and M. Teicher, On Galois covers of Hirzebruch surfaces, *Math. Ann.* **305** (1996) 493–539.

[11] B. Moishezon and M. Teicher, Simply connected algebraic surfaces of positive index, *Invent. Math.* **89** (1987) 601–643.

[12] L. H. Rowen, M. Teicher and U. Vishne, Coxeter covers of the symmetric groups, *J. Group Theory* **8** (2005) 139–169.
[13] V. Sergiescu, Graphes planaires et prsentations des groupes de tresses [Planar graphs and presentations of braid groups], *Math. Z.* **214**(3) (1993) 477–490 (in French).
[14] M. Teicher, On the quotient of the braid group by commutators of transversal half-twists and its group actions, *Topology Appl.* **78**(1–2) (1997) 153–186. Special issue on braid groups and related topics (Jerusalem, 1995).