TRIVIALITY OF THE FUNCTOR Coker($\mathrm{K}_1(F) \to \mathrm{K}_1(D)$) FOR DIVISION ALGEBRAS

ROOZBEH HAZRAT AND UZI VISHNE

Abstract. Let D be a division algebra with centre F. Consider the group $\mathrm{CK}_1(D) = D^*/F^*D'$ where D^* is the group of invertible elements of D and D' is its commutator subgroup. In this note we shall show that, assuming a division algebra D is a product of cyclic algebras, the group $\mathrm{CK}_1(D)$ is trivial if and only if D is an ordinary quaternion algebra over a real Pythagorean field F. We also characterize the cyclic central simple algebras with trivial CK_1, and show that CK_1 is not trivial for division algebras of index 4. Using valuation theory, the group $\mathrm{CK}_1(D)$ is computed for some valued division algebras.

1. Introduction

Let A be a local ring with centre R, a commutative local ring. Consider the functor $\mathrm{CK}_1(A) = \mathrm{Coker}(\mathrm{K}_1(R) \xrightarrow{i} \mathrm{K}_1(A))$ where i is the inclusion map. Thanks to the Dieudonné determinant for local rings, one can see that $\mathrm{CK}_1(A) = A^*/R^*A'$ where A^* and R^* are the groups of invertible elements of A and R respectively, and A' is the derived subgroup of A^*. If A is in addition an Azumaya algebra, then one can show that the group $\mathrm{CK}_1(A)$ is an Abelian group annihilated by n, where n^2 is the rank of A over R [5]. A study of this group in the case of central simple algebras is initiated in [7] and further in [6]. It has been established that despite of a "different nature" of this group from the reduced Whitehead group SK_1, the two groups have similar functorial properties. In [7] this functor is determined for tame and totally ramified division algebras over Henselian fields, and in particular for any finite Abelian group H, a division algebra D is constructed such that $\mathrm{CK}_1(D) = H \times H$. Further in [6], this functor is studied in more cases and examples of cyclic CK_1 (even over non local fields) are constructed. Our purpose in this paper is to address the conjecture raised in [7], that CK_1 can be trivial only if the index of the division algebra is 2. We show that if $\mathrm{CK}_1(A)$ is trivial where the central simple algebra A is a tensor product of cyclic algebras, then A is similar

Date: Jul. 17, 2004.

in the Brauer group to a cyclic algebra (Proposition 2.9). We characterize cyclic algebras with trivial CK_1 as split algebras or matrices over $(\frac{-1,-1}{F})$ (Theorem 2.10), and conclude that the conjecture holds for division algebras which are products of cyclic algebras (Theorem 2.12). In particular for a cyclic division algebra D, if $\mathrm{CK}_1(D)$ is trivial then D is an ordinary quaternion algebra and the centre of D is a real Pythagorean field.

Along the same lines, we show that $\mathrm{CK}_1(D)$ cannot be trivial if D is a division algebra of index 4, and furthermore if $\exp(\mathrm{CK}_1(D)) = 2$ then D decomposes as a product of two quaternion subalgebras. From the theorem mentioned above, it follows that if D is a cyclic division algebra of index p, an odd prime, then the exponent of $\mathrm{CK}_1(D)$ is exactly p. By exhibiting an example of a cyclic division algebra D of index $2p$ such that the exponent of $\mathrm{CK}_1(D)$ is p, we show that the converse is not true. It is not clear what conditions would be imposed on the algebraic structure of D if $\exp(\mathrm{CK}_1(D)) < \mathrm{ind}(D)$.

2. Triviality of CK_1

We study CK_1 together with a closely related functor. Let A be a central simple (finite dimensional) algebra over a field F, and set

$$\mathrm{NK}_1(A) = A^*/F^*A^{(1)}$$

where $A^{(1)}$ denotes the kernel of the reduced norm. Since $A' \subseteq A^{(1)}$, $\mathrm{NK}_1(A)$ is a quotient group of $\mathrm{CK}_1(A) = A^*/F^*A'$. In particular the triviality of NK_1 is a weaker assumption than that of CK_1. Obviously if $\mathrm{SK}_1(A) = A^{(1)}/A'$ is trivial, then $\mathrm{CK}_1(A) = \mathrm{NK}_1(A)$. We note one special case:

Remark 2.1. For split algebras, $A = \mathrm{M}_n(F)$, $\mathrm{CK}_1(A) = \mathrm{NK}_1(A)$ with the exception of $|F| = n = 2$ ($\mathrm{SL}_n(F)$ is the commutator subgroup of $\mathrm{GL}_n(F)$ except for this case).

The reduced norm induces an isomorphism

$$\mathrm{NK}_1(A) \cong \mathrm{Nrd}_A(A^*)/F^{*n} \tag{1}$$

where $n = \deg(A)$. In particular,

Remark 2.2. The triviality of $\mathrm{NK}_1(A)$ (in particular of $\mathrm{CK}_1(A)$) implies

$$\mathrm{Nrd}_A(A^*) = F^{*n}. \tag{2}$$

In turn, by definition of the reduced norm, Equation (2) holds if and only if

$$\mathrm{N}_{K/F}(K^*) = F^{*n} \tag{3}$$

for every separable maximal commutative subalgebra K of A.

It is obvious that $\mathrm{NK}_1(A)$, which is isomorphic to a subgroup of F^*/F^{*n}, is Abelian of exponent dividing n. For completeness, we sketch the argument showing $\exp(\mathrm{CK}_1(D)) \mid n$, for a division algebra of index n. Consider the sequence

$$\mathrm{K}_1(D) \xrightarrow{\mathrm{Nrd}_D} \mathrm{K}_1(F) \xrightarrow{i} \mathrm{K}_1(D) \tag{4}$$

where i is the inclusion map. One can see that the composition $i \circ \mathrm{Nrd}_D$ is equal to the exponentiation map η_n, defined by $\eta_n(a) = a^n$ (see for example the proof of [2, Lemma 4, p. 157]). From this it follows that for every $a \in D^*$, $a^n = \mathrm{Nrd}_D(a)c_a$ for some $c_a \in D'$, and so $a^n \equiv 1 \pmod{F^*D'}$.

If D is a division algebra and $A = \mathrm{M}_t(D)$, then using Dieudonné determinant one sees that $\mathrm{CK}_1(A) \cong D^*/F^{*t}D'$. Similarly one can show that $A^*/F^*A^{(1)} \cong D^*/F^{*t}D^{(1)}$.

Remark 2.3. With $A = \mathrm{M}_t(D)$ a central simple algebra,

$$\exp(\mathrm{CK}_1(D)) \mid \exp(\mathrm{CK}_1(A)) \mid t \cdot \exp(\mathrm{CK}_1(D))$$

and

$$\exp(\mathrm{NK}_1(D)) \mid \exp(\mathrm{NK}_1(A)) \mid t \cdot \exp(\mathrm{NK}_1(D)).$$

We will use the following property of NK_1:

Proposition 2.4. *Let A and B be central simple algebras of co-prime degrees. If $\mathrm{NK}_1(A \otimes B) = 1$ then $\mathrm{NK}_1(A) = \mathrm{NK}_1(B) = 1$.*

Proof. Let $n = \deg(A)$ and $m = \deg(B)$. If $a \in A^*$, then $\mathrm{Nrd}_A(a)^m = \mathrm{Nrd}_{A\otimes B}(a \otimes 1) \in F^{*nm} \subseteq F^{*n}$ by assumption, so $\mathrm{Nrd}_A(a)^m$ is trivial modulo F^{*n}. But the exponent of F^*/F^{*n} divides n which is prime to m, so $\mathrm{Nrd}_A(a)$ is trivial too. ☐

A stronger version of this holds for division algebras:

Theorem 2.5 ([6])**.** *Let A and B be central division algebras of co-prime indices over F. Then $\mathrm{CK}_1(A \otimes_F B) \cong \mathrm{CK}_1(A) \times \mathrm{CK}_1(B)$.*

The reduced Whitehead group is known to have a similar property. As noted in [6], the same result holds for NK_1.

Now assume Q is a quaternion division algebra over F, then Q has a maximal separable subfield K, with $\mathrm{Gal}(K/F) = \{1, \sigma\}$, such that $Q \cong K[j \mid j^2 = b,\ jkj^{-1} = \sigma(k)]$ for some element $b \in F^*$. If $\mathrm{char} F \neq 2$ then $K = F[i]$ where $i^2 = a \in F^*$, and $ji = -ij$. Any element of Q has the form $c_0 + c_1 i + c_2 j + c_3 ij$ $(c_0, \ldots, c_3 \in F)$, and the norm function is the quadratic form $\mathrm{Nrd}_Q(c_0 + c_1 i + c_2 j + c_3 ij) = c_0^2 - ac_1^2 - bc_2^2 +$

abc_3^2. One obtains a similar (though non-diagonal) quadratic form in characteristic 2. This argument provides an easy proof of the following special case of Wang's theorem [14].

Remark 2.6. For quaternion algebras over F, $Q^{(1)} = Q'$, and therefore $\mathrm{CK}_1(Q) = \mathrm{NK}_1(Q)$ (except for the case $|F| = 2$).

Proof. For division algebras, this follows from Hilbert theorem 90 for the separable subfields of Q, which are of course cyclic, and the fact that the norm of a non-separable element equals its square. The split case is Remark 2.1. $\square$

By Equation (1), $\mathrm{CK}_1(Q) \cong \mathrm{Nrd}(Q^*)/F^{*2}$. It follows that $|\mathrm{CK}_1(Q)|$ is the number of square classes in F^*/F^{*2} which are covered by the norm form. In particular, $\mathrm{CK}_1(Q) = 1$ if and only if the reduced norm of every element is a square.

For the next proposition, recall that F is real Pythagorean if $-1 \notin F^{*2}$ and sum of any two square elements is a square in F. It follows immediately that F is an ordered field.

Proposition 2.7. *Let Q be a quaternion division algebra. Then $\mathrm{CK}_1(Q)$ is trivial if and only if $Q = (\frac{-1,-1}{F})$ and F is Pythagorean.*

Proof. Assume $\mathrm{CK}_1(Q)$ is trivial. Write $Q = K[j]$ with $j^2 = b \in F^*$ as above, then $-b = \mathrm{N}_{F[j]/F}(j) = \mathrm{Nrd}_Q(j) \in F^{*2}$. Multiplying j by a suitable central element we may assume $b = -1$. If $\mathrm{char}F = 2$ then $b = 1$ and the algebra splits. Otherwise, $Q = (\frac{a,b}{F})$ and the same argument applies for a; therefore $Q = (\frac{-1,-1}{F})$, and we are done by the next proposition. $\square$

Proposition 2.8. *Let F be an arbitrary field. The following are equivalent.*

1) F is a real Pythagorean field.

2) $(\frac{-1,-1}{F})$ is a division algebra and $\mathrm{CK}_1(\frac{-1,-1}{F})$ is trivial.

3) $(\frac{-1,-1}{F})$ is a division algebra and every maximal subfield of $(\frac{-1,-1}{F})$ is F-isomorphic to $F(\sqrt{-1})$.

Proof. We shall show that 1) and 2) are equivalent. The equivalence of 1) and 3) is known (see [3]). Note that the definition implies that real Pythagorean fields have characteristic not 2.

1) $\Rightarrow$ 2) Suppose F is real Pythagorean. It is easy to see that $Q = (\frac{-1,-1}{F})$ is a division algebra. Now for any $x \in Q^*$, $\mathrm{Nrd}_Q(x)$ is a sum of four squares, thus $\mathrm{Nrd}_Q(Q^*) = F^{*2}$. As noted above, this equality forces CK_1 to be trivial.

2) $\Rightarrow$ 1) Since $Q = (\frac{-1,-1}{F})$ is a division ring, $-1 \notin F^{*2}$. The sum of two squares is a square since ${f_1}^2 + {f_2}^2 = \mathrm{Nrd}_Q(f_1 + f_2 i) \in F^{*2}$. If $-1 = {f_1}^2 + \cdots + {f_r}^2$ with r minimal, this shows $r = 1$, a contradiction. $\square$

F is called Euclidean if F^{*2} is an ordering of F. Over such fields, the only quaternion division algebra is the ordinary one, and from the above proposition it follows that its CK_1 is trivial.

We shall show that if a division algebra D is a product of cyclic algebras and has trivial CK_1, then it must be the ordinary quaternion algebra over a real Pythagorean field. We mention that there are examples of infinite dimensional division rings D such that D^* coincides with D' [9]. In the finite dimensional case, it is not hard to see that $D^* \neq D'$, in fact $\mathrm{K}_1(D) = D^*/D'$ is torsion free. However, essentially nothing is known in the case of algebraic (infinite dimensional) division rings.

Proposition 2.9. *Let $A = C_1 \otimes_F \ldots \otimes_F C_t$ be a central simple algebra, where $C_1, \ldots, C_t$ are cyclic algebras over F. If $\mathrm{NK}_1(A)$ is trivial, then A is similar in the Brauer group to a cyclic algebra of degree $\mathrm{lcm}(\deg C_1, \ldots, \deg C_t)$.*

Proof. By Proposition 2.4, and the fact that a tensor product of cyclic algebras of co-prime degrees is again cyclic, we may assume $\deg(A)$ is a prime power. We may assume $t > 1$. Let $n_i = \deg(C_i)$, and $n = \deg(A)$. For each i, let K_i be a cyclic maximal subfield of C_i, and $z_i \in C_i$ an element inducing an automorphism σ_i of order n_i of K_i/F. Then $b_i = z_i^{n_i} \in F^*$. Now, $\mathrm{Nrd}_A(z_i) = \mathrm{Nrd}_{C_i}(z_i)^{n/n_i} = ((-1)^{n_i - 1} b_i)^{n/n_i} = b_i^{n/n_i}$, where the last equality follows since n_i and n/n_i have the same parity. Now by Remark 2.2, b_i^{n/n_i} is an n-power in F^*, so (multiplying z_i by a central element) we may assume b_i is an n/n_i-root of unity. Taking a generator ρ of the group $\langle b_1, \ldots, b_t \rangle$, every C_i is a cyclic algebra of the form $(K_i/F, \sigma_i, \rho^{g_i})$ for some g_i, and their tensor product is similar in the Brauer group to a cyclic algebra of degree $\mathrm{lcm}(n_1, \ldots, n_t)$, as asserted. $\square$

Theorem 2.10. *Let A be a cyclic central simple algebra of prime power degree over F. Then $\mathrm{CK}_1(A) = 1$ if and only if $\mathrm{NK}_1(A) = 1$, if and only if one of the following options hold:*

1. *$A = \mathrm{M}_n(F)$, and every element of F is an n-power.*
2. *$A = (\frac{-1,-1}{F})$ and F is Pythagorean.*
3. *A is a matrix algebra of degree 2^t over $(\frac{-1,-1}{F})$, $t \geq 1$, and F is Euclidean.*

Proof. Assume $\mathrm{NK}_1(A) = 1$. Let $n = \deg(A)$. Let K be a maximal cyclic subfield of A, and $z \in A$ an element inducing an automorphism $\sigma \in \mathrm{Gal}(K/F)$ of order n. Then $b = z^n \in F^*$, and $\mathrm{Nrd}_A(z) = (-1)^{n-1}b$ is an n-power in F, by assumption. Multiplying z by a central element, we may assume $b = (-1)^{n-1}$.

If n is odd or $\mathrm{char}F = 2$, then $A = (K, \sigma, 1)$ splits. We may now assume n is a power of 2, and $b = -1$. If $n = 2$ then by Proposition 2.7 A splits, or $A = (\frac{-1,-1}{F})$ with F Pythagorean. Thus we may assume $n \geq 4$.

Let $L = K^{\sigma^2}$ be the quadratic subfield of K, and let $\alpha \in L$ be a generator such that $\sigma(\alpha) = -\alpha$ and $\alpha^2 \in F$. Since $\mathrm{Nrd}_A(\alpha) = \mathrm{N}_{L/F}(\alpha)^{n/2} = (-1)^{n/2}\alpha^n = \alpha^n$ is an n-power in F, we may assume $\alpha^n = 1$. Since L is a field, $\alpha^2 \neq 1$ so L has a primitive fourth root of unity, which we will denote by i.

Now $L[z^{n/2}]$, which is a commutative subalgebra of A, contains the idempotent $e = \frac{1}{2}(1 + iz^{n/2})$. Let $K' = K^{\sigma^{n/2}}$. Let $C = \mathrm{C}_A(F[e])$ be the centralizer in A, then $K'e$ is a cyclic subfield of Ce, of dimension $n/2$ over the center Fe, so Ce is a cyclic algebra of degree $n/2$ over Fe. But $C = eAe + (1-e)A(1-e)$ by Peirce decomposition, so $Ce = eAe$ is Brauer equivalent to A and $A \cong \mathrm{M}_2(Ce)$ by dimension consideration. The triviality of $\mathrm{NK}_1(A)$ implies triviality of $\mathrm{NK}_1(Ce)$ (Remark 2.3), so by induction on the degree we conclude that the underlying division algebra is either F or $D = (\frac{-1,-1}{F})$.

It remains to conclude the properties of F. If $A = \mathrm{M}_n(F)$, then $\mathrm{NK}_1(A) = \mathrm{CK}_1(A) \cong F^*/F^{*n}$ so the assumption is equivalent to $F^* = F^{*n}$. If $A = (\frac{-1,-1}{F})$ we are done by Proposition 2.7. Finally assume A is a proper matrix algebra over $D = (\frac{-1,-1}{F})$ which does not split. By assumption $A = \mathrm{M}_{n/2}(D)$ where $n \geq 4$ is a power of 2. Thus $\mathrm{Nrd}_A(A^*) = \mathrm{Nrd}_D(D^*) = \{a^2 + b^2 + c^2 + d^2 \,|\, a, b, c, d \in F\}$, clearly containing F^{*n}. But $F^{*n} \subseteq F^{*2}$, so we have an equality $\mathrm{Nrd}_A(A^*) = F^{*n}$ iff $F^{*2} + F^{*2} = F^{*2}$ and $F^{*2} = F^{*4}$. The latter equality is equivalent to $F^* = F^{*2} \cup -F^{*2}$, so $\mathrm{Nrd}_A(A^*) = F^{*n}$ iff F is Euclidean. □

Taking prime-power decomposition, we obtain

Corollary 2.11. *Let A be a cyclic central simple algebra over F with trivial $\mathrm{NK}_1(A)$. Then A is a matrix algebra over F or over $(\frac{-1,-1}{F})$.*

Theorem 2.12. *Let D be a division algebra which is a tensor product of cyclic algebras. Then $\mathrm{CK}_1(D) = 1$ if and only if $\mathrm{NK}_1(D) = 1$, if and only if D is an ordinary quaternion division algebra over a real Pythagorean field.*

Proof. If F is a real Pythagorean field and D is the quaternion algebra over F then $\mathrm{CK}_1(D) = 1$ by Proposition 2.7.

Now suppose $\mathrm{NK}_1(D) = 1$. We may decompose $D \cong D_1 \otimes \ldots \otimes D_r$ for D_i division algebras of prime power degree, each D_i being a tensor product of cyclic algebras. If D_i is the tensor product of $t > 1$ cyclic algebras, then by Proposition 2.9 it is similar to a cyclic algebra of smaller degree, contradicting the assumption that D_i is a division algebra. Thus, D_i is cyclic, and by the previous theorem D_i is either F or the standard quaternions. □

Remark 2.13. 1. Theorem 2.12 gives a criterion for a division algebra not to be a product of cyclic algebras. In particular if D has odd prime index the triviality of $\mathrm{CK}_1(D)$ would imply D is non-cyclic.

2. By Merkurjev-Suslin theorem every central simple algebra is similar to a tensor product of cyclic algebras, if the center has enough roots of unity. However, the triviality of $\mathrm{CK}_1(D)$ does not imply the triviality of $\mathrm{CK}_1(A)$ for $A = \mathrm{M}_t(D)$. Indeed, let $D = (\frac{-1,-1}{F})$ over a Pythagorean field F, and let $A = \mathrm{M}_t(D)$. Then $\mathrm{CK}_1(A) \cong \mathrm{Nrd}(D^*)/F^{*2t} = F^{*2}/F^{*2t}$, which is not trivial in general (e.g. if t is even and F is not Euclidean).

Theorem 2.14. *Let D be a division algebra of index* 4. *If* $\mathrm{NK}_1(D)$ *has exponent* ≤ 2, *then D is decomposable.*

Proof. By Albert's theorem [1, Thm. XI.9], D is a crossed product with respect to $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Let K/F be a maximal subfield of D with Galois group $\langle \sigma_1, \sigma_2 \rangle \cong G$, and let $z_1, z_2 \in D$ be elements inducing the automorphisms σ_1, σ_2 on K, respectively. Let $K_i = K^{\sigma_i}$ denote the fixed subfields. As in Remark 2.2, the assumption $D^{*2} \subseteq F^* D'$ implies that for every $u \in D^*$, $\mathrm{Nrd}_D(u)^2 \in F^{*4}$, or equivalently $\mathrm{Nrd}(u) \in \pm F^{*2}$.

Since the reduced norm is multiplicative, $\mathrm{Nrd}(z) \in F^{*2}$ for at least one of the elements $z \in \{z_1, z_2, z_1 z_2\}$. Changing names of the generators of $\mathrm{Gal}(K/F)$ if necessary, we may assume $\mathrm{Nrd}(z_1) \in F^{*2}$. Let $b_1 = z_1^2$, which is an element of K_1. The field $K_1[z_1]$ is a maximal subfield as $z_1 \notin K_1$, and

$$\begin{aligned} \mathrm{Nrd}_D(z_1) &= \mathrm{N}_{K_1[z_1]/F}(z_1) = \mathrm{N}_{K_1/F}\mathrm{N}_{K_1[z_1]/K_1}(z_1) \\ &= \mathrm{N}_{K_1/F}(-z_1^2) = \mathrm{N}_{K_1/F}(b_1). \end{aligned}$$

It follows that $\mathrm{N}_{K_1/F}(f^{-1}b_1) = 1$ for some $f \in F^*$. Therefore there is an element $c \in K_1$ such that $b_1 = f\sigma_2(c)c^{-1}$, and then $(cz_1)^2 = c^2 b_1 = fc\sigma_2(c) = f\mathrm{N}_{K_1/F}(c) \in F$. Since cz_1 induces a non-trivial automorphism on K_2, $Q = K_2[cz_1]$ is a quaternion subalgebra of D, which is thus the product of Q and its centralizer. □

Corollary 2.15. *Let D be a division algebra of index 4, then* $\mathrm{NK}_1(D)$ *is non-trivial, and in particular* $\mathrm{CK}_1(D)$ *is non-trivial.*

Proof. If $\mathrm{NK}_1(D) = 1$ then by the last theorem D is isomorphic to a product of quaternions, and the result follows from Theorem 2.12. □

3. Examples

The precise connection between $\exp(\mathrm{CK}_1(D))$ and the index of D is not clear. We demonstrate the situation with algebras of index 4 (where by Theorem 2.14, $\exp(CK(D)) < 4$ implies decomposability).

Example 3.1. A (non-cyclic) decomposable division algebra of index 4 can have $\exp(\mathrm{CK}_1(D))$ either 2 or 4. Indeed, let $F = \mathbb{R}(x_1, x_2, x_3, x_4)$ and consider

$$D = \left(\frac{x_1, x_2}{F}\right) \otimes_F \left(\frac{x_3, x_4}{F}\right).$$

Let $z_1, z_3 \in D$ be commuting elements such that $z_1^2 = x_1$ and $z_3^2 = x_3$, then $\mathrm{Nrd}_D(1+z_1+z_3) = \mathrm{N}_{F[z_1,z_3]/F}(1+z_1+z_3) = 1-2(x_1+x_3)+(x_1-x_3)^2$, which is not a square in F^*, and so its class in F^*/F^{*4} has order 4, and $\exp(\mathrm{CK}_1(D)) = 4$. By considering the norms of the elements $\alpha + z_1 + z_3$ ($\alpha \in \mathbb{R}$), it is easy to show that $|\mathrm{CK}_1(D)| = \infty$.

Now let $\bar{F} = \mathbb{R}((x_1))((x_2))((x_3))((x_4))$, a Henselian field. Consider

$$D \otimes_F \bar{F} = \left(\frac{x_1, x_2}{\bar{F}}\right) \otimes_F \left(\frac{x_3, x_4}{\bar{F}}\right).$$

This is a tame and totally ramified division algebra with relative group $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. In [6] it was shown that CK_1 of a tame and totally ramified division algebra over a Henselian field, is isomorphic to the relative value group. Thus $\exp(\mathrm{CK}_1(D \otimes_F \bar{F})) = 2$. However notice that F is not Henselian, so $\mathrm{CK}_1(D)$ is not determined in [6] (even though D is totally ramified with respect to the valuation restricted from $D \otimes_F \bar{F}$). The division algebra $D \otimes_F \bar{F}$ is non-cyclic and if we add a root of unity of order 4 to the base field, then $\mathrm{SK}_1 \neq 1$. This was noticed for the first time by Draxl [2, p.168-169].

Finally, $\exp(\mathrm{CK}_1(D \otimes_F \bar{F}(y)))$ is again 4 when y is transcendental over F, as the next proposition shows. In particular, extension of scalars may either increase or decrease the exponent of CK_1.

Proposition 3.2. *Let D be an F-central division algebra of index n, and y be an independent indeterminate over F. Then* $\exp(\mathrm{CK}_1(D(y))) = n$*, where $D(y) = D \otimes_F F(y)$.*

Proof. Consider the element $y - a \in D(y)$ where $a \in D$. It can be seen that

$$\mathrm{Nrd}_{D(y)}(a-y) = \mathrm{Chr}_D(a),$$

where $\mathrm{Chr}_D(a)$ is the reduced characteristic polynomial of a in D. But $\mathrm{Chr}_D(a) = f(y)^{n/m}$ where $f(y)$ is the minimal polynomial of a and m is the degree of $f(y)$. Then, the order of $\mathrm{Nrd}_{D(y)}(a-y) = f(y)^{n/m}$ in the quotient group

$$\frac{D(y)^*}{F(y)^* D(y)^{(1)}} \cong \frac{\mathrm{Nrd}_{D(y)}(D(y)^*)}{F(y)^{*n}}$$

equals m, and we are done by choosing a that generates a maximal subfield of D. □

We recall from [6] that if D is a tame and totally ramified division algebra over a Henselian field, then $\exp(\mathrm{CK}_1(D)) = \mathrm{ind}(D)$ if and only if D is cyclic. In fact from Theorem 2.12 it follows that if D is a cyclic division algebra of index p, an odd prime, then the exponent of $\mathrm{CK}_1(D)$ is exactly p. On the other hand, we now present an example of a cyclic decomposable F-division algebra D of index $2p$, p an odd prime, with a proper F-division subalgebra $A \subset D$, where $\mathrm{CK}_1(A) \cong \mathrm{CK}_1(D)$. In particular $\exp(\mathrm{CK}_1(D)) < \mathrm{ind}(D)$ even though D is cyclic, unlike the situation for totally ramified algebras of prime index. (This example also shows that $\exp(\mathrm{CK}_1)$ does not follow the same pattern as $\exp(D)$).

For this we need the Fein-Schacher-Wadsworth example of a division algebra of index $2p$ over a Pythagorean field F [4]. We briefly recall the construction. Let p be an odd prime and K/F be a cyclic extension of dimension p of real Pythagorean fields, and let σ be a generator of $\mathrm{Gal}(K/F)$. Then $K((x))/F((x))$ is a cyclic extension where $K((x))$ and $F((x))$ are the Laurent series fields of K and F, respectively. The algebra

$$D = \left(\frac{-1,-1}{F((x))}\right) \otimes_{F((x))} \Big(K((x))/F((x)), \sigma, x\Big)$$

was shown to be a division algebra of index $2p$. Since F is real Pythagorean, so is $F((x))$. Now by Theorem 2.5,

$$\mathrm{CK}_1(D) \cong \mathrm{CK}_1\left(\frac{-1,-1}{F((x))}\right) \times \mathrm{CK}_1(A)$$

where $A = \big(K((x))/F((x)), \sigma, x\big)$. By Proposition 2.8, $\mathrm{CK}_1(\left(\frac{-1,-1}{F((x))}\right)) = 1$, so $\mathrm{CK}_1(D) \cong \mathrm{CK}_1(A)$ and has exponent p (Theorem 2.12).

We end this note with a remark on the computation of CK_1.

Remark 3.3. 1. Some notions from the theory of quadratic forms, like rigidity of an element, which plays a role in the study of the extensions of Pythagorean fields, can be formulated as properties of the group CK_1. Recall that $a \in F$ is called *rigid* if $a \notin \pm F^{*2}$ and $F^{*2} + aF^{*2} = F^{*2} \cup aF^{*2}$. If $K = F(\sqrt{a})$ is a quadratic extension of F, then K is real Pythagorean if and only if F is real Pythagorean and a is rigid (see [10, §5]). It is not difficult to see that if F is a real Pythagorean field and $a \notin \pm F^{*2}$, then a is rigid if and only if $\mathrm{CK}_1(\frac{-1,-a}{F}) = \mathbb{Z}/2$.

2. The group CK_1 is highly sensitive to the arithmetic of the ground field. Taking a field F with $-1 \notin F^{*2}$ and $\mathrm{char} F \neq 2$, $D = \left(\frac{x,x}{F((x))}\right)$ is a division algebra. For $F = \mathbb{R}$ we have that $\mathrm{CK}_1(D) \cong \mathbb{Z}/2$, whereas for $F = \mathbb{F}_q$ ($q \equiv 3 \pmod 4$), $\mathrm{CK}_1(D) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. These are examples of semiramified division algebras. In fact a quaternion division algebra could be unramified, semiramified or totally ramified, and one can compute the CK_1 of such algebras by means of valuation theory (cf. [6] and [13] for an excellent survey of the valuation theory of division algebras). For quaternions, one can alternatively use quadratic form techniques.

Acknowledgments. The authors thank the referee for helpful comments.

References

[1] A. A. Albert, Structure of Algebras, Amer. Math. Soc. Coll. Publ., Vol. XXIV, Providence, 1961.
[2] P. Draxl, Skew fields. London Mathematical Society Lecture Note Series **81**, Cambridge University Press, Cambridge, 1983.
[3] B. Fein, M. Schacher, *The ordinary quaternions over a Pythagorean field*, *Proc. Amer. Math. Soc.* **60** (1976), 16–18 (1977).
[4] B. Fein, M. Schacher, A. Wadsworth, *Division rings and the square root of* -1, *J. Algebra* **65**(2), (1980), 340–346.
[5] R. Hazrat, *Reduced* K*-theory for Azumaya algebras*, *preprint*
[6] R. Hazrat, SK_1*-like functors for division algebras*, *J. of Algebra* **239** (2001), 573–588.
[7] R. Hazrat, M. Mahdavi-Hezavehi, B. Mirzaii, *Reduced* K*-theory and the group* $G(D) = D^*/F^*D'$, Algebraic K-theory and its applications (H. Bass, Editor), 403–409, World Sci. Publishing, River Edge, NJ, 1999.
[8] N. Jacobson, Finite Dimensional Division Algebras over Fields, Springer, 1996.
[9] O. Kegel, *Zur Einfachheit der multiplikativen Gruppe eines existentiell abgeschlossenen Schiefkoerpers*, Results Math. **35**(1–2) (1999), 103–106.
[10] T. Y. Lam, Orderings, valuations and quadratic forms. CBMS Regional Conference Series in Mathematics **52**, American Mathematical Society, Providence, RI, 1983.
[11] J.-P. Tignol, A. Wadsworth, *Totally ramified valuations on finite-dimensional division algebras*, *Trans. Amer. Math. Soc.* **302**(1), (1987), 223–250.
[12] W. Scharlau, Quadratic and Hermitian forms, Springer-Verlag, Berlin, 1985.
[13] A. Wadsworth, *Valuation theory on finite dimensional division algebras*, Fields Inst. Commun. **32**, Amer. Math. Soc., Providence, RI, (2002), 385–449.
[14] S. Wang, *On the commutator group of a simple algebra*, Amer. J. Math **72**, (1950) 323–334.

Mathematical Sciences Institute, John Dedman Building (# 27), Australian National University, Canberra ACT 0200, Australia

Current address: Department of Pure Mathematics, Queen's University Belfast, Belfast BT7 1NN, Northern Ireland, U.K.

E-mail address: r.hazrat@qub.ac.uk

Department of Mathematics, Yale University, 10 Hillhouse Ave., New Haven, CT 06520, USA

Current address: Department of Mathematics, Bar Ilan University, Ramat Gan 52900, Israel

E-mail address: vishne@math.biu.ac.il