

# THE FUNDAMENTAL GROUP OF A GALOIS COVER OF $\mathbb{CP}^1 \times T$

MEIRAV AMRAM<sup>1</sup>, DAVID GOLDBERG<sup>2</sup>, MINA TEICHER AND UZI VISHNE<sup>3</sup>

ABSTRACT. Let  $T$  be the complex projective torus, and  $X$  the surface  $\mathbb{CP}^1 \times T$ . Let  $X_{\text{Gal}}$  be its Galois cover with respect to a generic projection to  $\mathbb{CP}^2$ . In this paper we compute the fundamental group of  $X_{\text{Gal}}$ , using the degeneration and regeneration techniques, the Moishezon-Teicher braid monodromy algorithm and group calculations. We show that  $\pi_1(X_{\text{Gal}}) = \mathbb{Z}^{10}$ .

## 1. OVERVIEW

Let  $T$  be a complex torus in  $\mathbb{CP}^2$ . We compute the fundamental group of the Galois cover with respect to a generic map of the surface  $X = \mathbb{CP}^1 \times T$  to  $\mathbb{CP}^2$ . We embed  $X$  into a projective space using the Segre map  $\mathbb{CP}^1 \times \mathbb{CP}^2 \rightarrow \mathbb{CP}^5$  defined by  $(s_0, s_1) \times (t_0, t_1, t_2) \mapsto (s_0t_0, s_1t_0, s_0t_1, s_1t_1, s_0t_2, s_1t_2)$ . Then, a generic projection  $f: X \rightarrow \mathbb{CP}^2$  is obtained by projecting  $X$  from a general plane in  $\mathbb{CP}^5 - X$  to  $\mathbb{CP}^2$ . The Galois cover can now be defined as the closure of the  $n$ -fold fibered product  $X_{\text{Gal}} = \overline{X \times_f \cdots \times_f X - \Delta}$  where  $n$  is the degree of the map  $f$ , and  $\Delta$  is the generalized diagonal. The closure is necessary because the branched fibers are excluded when  $\Delta$  is omitted.

The fundamental group  $\pi_1(X_{\text{Gal}})$  is related to the fundamental group of the complement of the branch curve. The latter is an important invariant of  $X$ , which can be used to classify algebraic surfaces of a general type, up to deformations. Such an invariant is finer than the famous Sieberg-Witten invariants and thus can serve as a tool to distinguish diffeomorphic surfaces which are not deformation of each other (see [4], [12] and [13]) a problem which is referred to as the Diff-Def problem. The algorithms and problems

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that arise in the computation of these two types of groups are related, and one hopes to be able to compute such groups for various types of surfaces.

Since the induced map  $X_{\text{Gal}} \rightarrow \mathbb{CP}^2$  has the same branch curve  $S$  as  $f: X \rightarrow \mathbb{CP}^2$ , the fundamental group  $\pi_1(X_{\text{Gal}})$  is related to  $\pi_1(\mathbb{CP}^2 - S)$ . In fact it is a normal subgroup of the quotient of  $\pi_1(\mathbb{CP}^2 - S)$  by the normal subgroup generated by the squares of the standard generators. In this paper we employ braid monodromy techniques, the van Kampen theorem and various computational methods of groups to compute a presentation for the quotient  $\tilde{\Pi}_1$  from which  $\pi_1(X_{\text{Gal}})$  can be derived. Our main result is that  $\pi_1(X_{\text{Gal}}) = \mathbb{Z}^{10}$  (Theorem 9.3).

The fundamental group of the Galois cover of the surface  $X = \mathbb{CP}^1 \times T$  is a step in computing the same group for  $T \times T$  [2], and will later appear in local computations of fundamental groups of the Galois cover of  $K3$ -surfaces.

It turns out that a property of the Galois covers that were treated before (see [6], [7] or [9]) is lacking in the Galois cover of  $X = \mathbb{CP}^1 \times T$ . In all the cases computed so far,  $X$  had the property that the fundamental group of the graph defined on the planes of the degenerated surface  $X_0$  by connecting every two intersecting planes, is generated by the cycles around the intersection points. Our surface, together with a parallel work on  $T \times T$  [2], are the first cases for which this assumption does not hold. The significance of this 'redundancy' property of  $S_0$  will be explained in Section 6 (and in more details in [2]).

The paper is organized as follows. In Section 2 we describe the degeneration of the surface  $X = \mathbb{CP}^1 \times T$  and the degenerated branch curve. In Sections 3 and 4 we study the regeneration of this curve and its braid monodromy factorization. We also get a presentation for  $\pi_1(\mathbb{C}^2 - S, u_0)$ , the fundamental group of the complement of the regenerated branch curve in  $\mathbb{C}^2$ , see Theorem 4.3. In Section 5 we present the homomorphism  $\psi: \tilde{\pi}_1(\mathbb{C}^2 - S, u_0) \rightarrow S_6$ , whose kernel is  $\pi_1(X_{\text{Gal}}^{\text{Aff}})$ . In Section 6 we study a natural Coxeter quotient of  $\pi_1(\mathbb{C}^2 - S, u_0)$ , and give its structure. In Sections 7 and 8 we use the Reidmeister-Schreier method to give a new presentation for  $\pi_1(X_{\text{Gal}}^{\text{Aff}})$ , see Theorem 8.10. In Section 9 we introduce the projective relation, and prove the main result about the structure of  $\pi_1(X_{\text{Gal}})$ .

## 2. DEGENERATION OF $\mathbb{CP}^1 \times T$

In the computation of braid monodromies it is often useful to replace the surface with a degenerated object, made of copies of  $\mathbb{CP}^2$ . It is easy to see that  $T$  degenerates to a triangle of complex projective lines (see [1, Subsection 1.6.3]), so  $X$  degenerates to a union of three quadrics  $Q_1, Q_2$ , and  $Q_3$ ,  $Q_i = \mathbb{CP}^1 \times \mathbb{CP}^1$ , which we denote by  $X_1$ , see Figure 1.

Each square in Figure 1 represents a quadric surface. Since  $T$  degenerates to a triangle,  $Q_1$  and  $Q_3$  intersect, so the left and right edges of  $X_1$  are identified. Therefore, we can view  $X_1$  as a triangular prism.

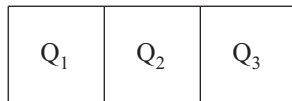


FIGURE 1. The space  $X_1$

Each quadric in  $X_1$  can be further degenerated to a union of two planes. In Figure 2 this is represented by a diagonal line which divides each square into two triangles, each one isomorphic to  $\mathbb{C}\mathbb{P}^2$ .

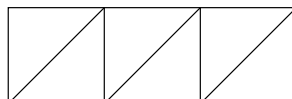


FIGURE 2. The simplicial complex  $X_0$

We shall refer to this diagram as the simplicial complex of  $X_0$ . A common edge between two triangles represents the intersection line of the two corresponding planes. The union of the intersection lines is the ramification locus in  $X_0$  of  $f_0: X_0 \rightarrow \mathbb{C}\mathbb{P}^2$ , denoted by  $R_0$ . Let  $S_0 = f_0(R_0)$  be the degenerated branch curve. It is a line arrangement, composed of all the intersection lines.

A vertex in the simplicial complex represents an intersection point of three planes. The vertices represent singular points of  $R_0$ . Each of these vertices is called a 3-point (reflecting the number of planes which meet there).

The vertices may be given any convenient enumeration. We have chosen left to right, bottom to top enumeration, see Figure 3. The extreme vertices are pairwise identified, as well as the left and right edges.

We create an enumeration of the edges based upon the enumeration of the vertices using reverse lexicographic ordering: if  $L_1$  and  $L_2$  are two lines with end points  $\alpha_1, \beta_1$  and  $\alpha_2, \beta_2$  respectively ( $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ ), then  $L_1 < L_2$  iff  $\beta_1 < \beta_2$ , or  $\beta_1 = \beta_2$  and  $\alpha_1 < \alpha_2$ . The resulting enumeration is shown in Figure 4. This enumeration dictates the order of the regeneration of the lines to curves, see the next section. The horizontal lines at the top and bottom do not represent intersections of planes and hence are not numbered.

We enumerate the triangles  $\{P_i\}_{i=1}^6$  also according to the enumeration of vertices in reverse lexicographic order. If  $P_i$  and  $P_j$  have vertices  $\alpha_1, \alpha_2, \alpha_3$

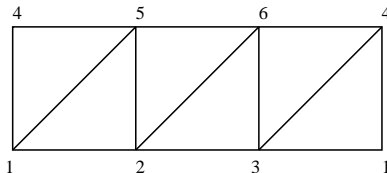


FIGURE 3

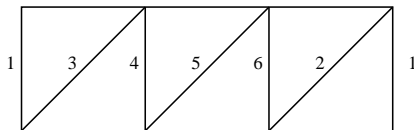


FIGURE 4

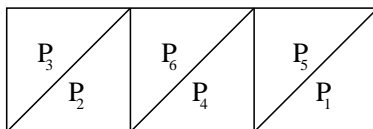


FIGURE 5

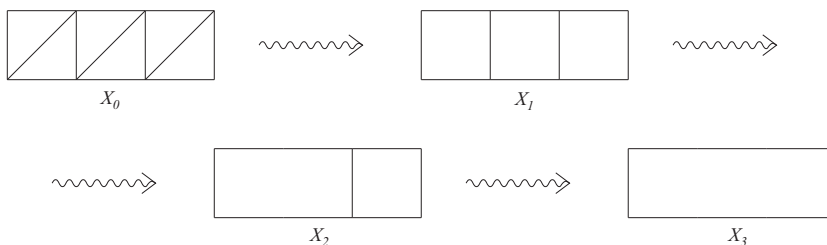


FIGURE 6. The regeneration process

and  $\beta_1, \beta_2, \beta_3$  respectively, with  $\alpha_1 < \alpha_2 < \alpha_3$  and  $\beta_1 < \beta_2 < \beta_3$ , then  $P_i < P_j$  iff  $\alpha_3 < \beta_3$ , or  $\alpha_3 = \beta_3$  and  $\alpha_2 < \beta_2$ , or  $\alpha_3 = \beta_3$ ,  $\alpha_2 = \beta_2$  and  $\alpha_1 < \beta_1$ . The enumeration is shown in Figure 5.

### 3. REGENERATION OF THE BRANCH CURVE

**3.1. The Braid Monodromy of  $S_0$ .** Starting from the branch curve  $S_0$ , we reverse the steps in the degeneration of  $X$  to regenerate the braid monodromy of  $S$ . Figure 6 shows the three steps to recover the original object  $X_3 = X$ .

Recall that  $X$  comes with an embedding to  $\mathbb{CP}^5$ . At each step of the regeneration, the generic projection  $\mathbb{CP}^5 \rightarrow \mathbb{CP}^2$  restricts to a generic map  $f_i: X_i \rightarrow \mathbb{CP}^2$ . Let  $R_i \subset X_i$  be the ramification locus of  $f_i$  and  $S_i \subset \mathbb{CP}^2$  the corresponding branch curve.

We have enumerated the six planes  $P_1, \dots, P_6$  which comprise  $X_0$ , the six intersection lines  $\hat{L}_1, \dots, \hat{L}_6$ , and their six intersection points  $\hat{V}_1, \dots, \hat{V}_6$ . Let  $L_i$  and  $V_j$  denote the projections of  $\hat{L}_i$  and  $\hat{V}_j$  to  $\mathbb{CP}^2$  by the map  $f_0$ . Clearly  $R_0 = \bigcup_{i=1}^6 \hat{L}_i$  and  $S_0 = \bigcup_{i=1}^6 L_i$ . Let  $C$  be the line arrangement consisting of all lines through pairs of the  $V_j$ s. The degenerated branch curve  $S_0$  is a subarrangement of  $C$ . Since  $C$  is a dual to generic arrangement, Moishezon's results in [5] (and later on Theorem IX.2.1 in [8]) gives us a braid monodromy factorization for  $C$ :  $\Delta_C^2 = \prod_{j=1}^6 C_j \Delta_j^2$  where  $\Delta_j^2$  is the monodromy around

$V_j$  and the  $C_j$  consist of products of the monodromies around the other intersection points of  $C$ . This factorization can be restricted to  $S_0$  by removing from the braids all strands which correspond to lines of  $C$  that do not appear in  $S_0$ , and deleting all factors which correspond to intersections in  $C$  that do not appear in  $S_0$ . Thus we get a braid monodromy factorization:  $\Delta_{S_0}^2 = \prod_{j=1}^6 \tilde{C}_j \tilde{\Delta}_j^2$ . The  $\tilde{C}_j$  and  $\tilde{\Delta}_j^2$  and their regenerations are formulated more precisely in the following subsections.

**3.2.  $\tilde{\Delta}_j^2$  and its Regeneration.** Consider an affine piece of  $S_0 \subset \mathbb{CP}^2$  and take a generic projection  $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}$ . Let  $N$  be the set of the projections of the singularities and branch points with respect to  $\pi$ . Choose  $u \in \mathbb{C} - N$  and let  $\mathbb{C}_u = \pi^{-1}(u)$  be a generic fiber.

A path from a point  $j$  to a point  $k$  below the real line is denoted by  $\underline{Z}_{jk}$ , and the corresponding halftwist by  $\underline{Z}_{jk}$ .

Two lines  $\hat{L}_j$  and  $\hat{L}_k$  which meet in  $X_0$  give rise to braids connecting  $j = L_j \cap \mathbb{C}_u$  and  $k = L_k \cap \mathbb{C}_u$ , namely a fulltwist  $\underline{Z}_{jk}^2$  of  $j$  and  $k$ . This is done in the following way: let  $V_i = L_j \cap L_k$  be the intersection point, then  $\tilde{\Delta}_i^2 = \underline{Z}_{jk}^2$ .

We shall analyze the regeneration of the local braid monodromy of  $S$  in a small neighborhood of each  $V_i$ . The case of non-intersecting lines (which give 'parasitic intersections') is discussed in the next subsection.

The degenerated branch curve  $S_0$  has six singularities coming from the 3-points of  $R_0$ , shown in Figure 7.

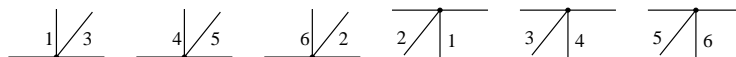


FIGURE 7. Enumeration around the 3-points

Each pair of lines intersecting at a 3-point regenerates in  $S_1$  (the branch curve of  $X_1$ ) as follows: the diagonal line becomes a conic, and the vertical line is tangent to it. In the next step of the regeneration the point of tangency becomes three cusps according to the third regeneration rule (which was quoted in [5] and proven in [10, p.337]). This is enough information to compute  $H_{V_i}$ , the local braid monodromy of  $S$  in a neighborhood of  $V_i$ , see these specific computations for this case in [1, Subsection 1.10.4].

In the regeneration, each point  $\alpha$  on the typical fiber is replaced by two close points  $\alpha, \alpha'$ . Denote by  $\rho_\alpha = \underline{Z}_{\alpha\alpha'}$  the counterclockwise halftwist of  $\alpha$  and  $\alpha'$ .

The following table presents the global form of the local braid monodromies, as quoted in [5], and presents also the application of this global form to our case.

**Table 3.1.** *The local braid monodromies  $H_{V_i}$  are as follows. For every fixed  $i$ , let  $\alpha < \beta$  be the lines intersecting at  $V_i$ . Let  $Z_{\alpha\alpha',\beta}^3 = (\mathbf{Z}_{\alpha'\beta}^3)_{\rho_\alpha} \cdot \mathbf{Z}_{\alpha'\beta}^3 \cdot (\mathbf{Z}_{\alpha'\beta}^3)_{\rho_\alpha}^{-1}$  and  $Z_{\alpha',\beta\beta'}^3 = (\mathbf{Z}_{\alpha'\beta}^3)_{\rho_\beta} \cdot \mathbf{Z}_{\alpha'\beta}^3 \cdot (\mathbf{Z}_{\alpha'\beta}^3)_{\rho_\beta}^{-1}$ .*

*For  $i = 1, 2, 4$  we have  $H_{V_i} = \mathbf{Z}_{\alpha\alpha',\beta'}^3 \cdot \mathbf{Z}_{\beta\beta'(\alpha)}$ , where  $\mathbf{Z}_{\beta\beta'(\alpha)}$  is the halftwist corresponding to the path shown in Figure 8. For  $i = 3, 5, 6$  we have  $H_{V_i} = \mathbf{Z}_{\alpha',\beta\beta'}^3 \cdot \mathbf{Z}_{\alpha\alpha'(\beta)}$ , where  $\mathbf{Z}_{\alpha\alpha'(\beta)}$  is the halftwist corresponding to the path shown in Figure 9.*

*In particular we have*

$$\begin{aligned} H_{V_1} &= \mathbf{Z}_{11',3'}^3 \cdot \mathbf{Z}_{33'(1)}, \\ H_{V_2} &= \mathbf{Z}_{44',5'}^3 \cdot \mathbf{Z}_{55'(4)}, \\ H_{V_3} &= \mathbf{Z}_{2',66'}^3 \cdot \mathbf{Z}_{22'(6)}, \\ H_{V_4} &= \mathbf{Z}_{11',2'}^3 \cdot \mathbf{Z}_{22'(1)}, \\ H_{V_5} &= \mathbf{Z}_{3',44'}^3 \cdot \mathbf{Z}_{33'(4)}, \\ H_{V_6} &= \mathbf{Z}_{5',66'}^3 \cdot \mathbf{Z}_{55'(6)}. \end{aligned}$$

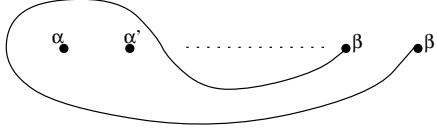


FIGURE 8.  $\mathbf{Z}_{\beta\beta'(\alpha)}$  for  $i = 1, 2, 4$

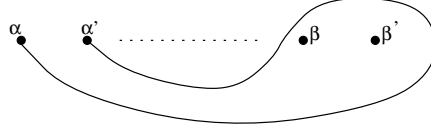


FIGURE 9.  $\mathbf{Z}_{\alpha\alpha'(\beta)}$  for  $i = 3, 5, 6$

The table given in Figure 10 presents the six monodromy factorization, one for every point  $V_1, \dots, V_6$ . For each point, the first path represents three factors obtained from cusps, and the other represents the fourth factor, obtained from the branch point (as shown in Figures 8 and 9). The relations obtained from these braids are given in Theorem 4.3.

**3.3.  $\tilde{C}_j$  and its Regeneration.** There are lines which do not meet in  $X_0$  but whose images meet in  $\mathbb{C}^2$ . Such an intersection is called a parasitic intersection. Each pair of disjoint lines  $\hat{L}_i$  and  $\hat{L}_j$  give rise to a certain fulltwist, see [8, Theorem IX.2.1]. This is denoted as  $\tilde{\mathbf{Z}}_{ij}^2$ , corresponding to a path  $\tilde{z}_{ij}$ , running from  $i$  over the points up to  $j_0$ , then under  $j_0$  up to  $j$ , where  $j_0$  is the least numbered line which shares the upper vertex of  $L_j$ .

pt.	the braid	exp.	the path representing the braid
$V_1$	$\rho_1^i \underline{\mathbf{Z}}_{1'3'} \rho_1^{-i} \quad i = 0, 1, 2$	3	
	$\mathbf{Z}_{33'}(1)$	1	
$V_2$	$\rho_4^i \underline{\mathbf{Z}}_{4'5'} \rho_4^{-i} \quad i = 0, 1, 2$	3	
	$\mathbf{Z}_{55'}(4)$	1	
$V_3$	$\rho_6^i \underline{\mathbf{Z}}_{2'6'} \rho_6^{-i} \quad i = 0, 1, 2$	3	
	$\mathbf{Z}_{22'}(6)$	1	
$V_4$	$\rho_1^i \underline{\mathbf{Z}}_{1'2'} \rho_1^{-i} \quad i = 0, 1, 2$	3	
	$\mathbf{Z}_{22'}(1)$	1	
$V_5$	$\rho_4^i \underline{\mathbf{Z}}_{3'4'} \rho_4^{-i} \quad i = 0, 1, 2$	3	
	$\mathbf{Z}_{33'}(4)$	1	
$V_6$	$\rho_6^i \underline{\mathbf{Z}}_{5'6'} \rho_6^{-i} \quad i = 0, 1, 2$	3	
	$\mathbf{Z}_{55'}(6)$	1	

FIGURE 10. Monodromy factorizations

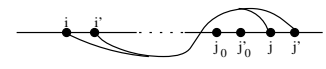
factor	corresponding paths	singularity types	degrees
$\tilde{\mathbf{Z}}_{ii',jj'}$		four nodes	2,2,2,2

FIGURE 11

As discussed in [7] and [1] the degree of the regenerated branch curve  $S$  is twice the degree of  $S_0$ . Consequently each line  $L_i \subset S_0$  divides locally into two branches of  $S$  and each  $i = L_i \cap \mathbb{C}_u$  divides into two points,  $i$  and  $i'$ . According to the second regeneration rule (quoted in [5] and proved in [10, p. 337]) the fulltwist  $\tilde{\mathbf{Z}}_{ij}^2$  becomes  $\tilde{\mathbf{Z}}_{ii',jj'}^2$ , which compounds four nodes of  $S$ , namely  $\tilde{\mathbf{Z}}_{ij}^2$ ,  $\tilde{\mathbf{Z}}_{i'j'}^2$ ,  $\tilde{\mathbf{Z}}_{i'j}^2$  and  $\tilde{\mathbf{Z}}_{ij'}^2$  as shown in Figure 11. These are the factors in the regenerations  $C'_j$  of the  $\tilde{C}_j$ . In Table 3.2 we construct the paths which correspond to these braids in our case.

Pick a base point  $u_0 \in \mathbb{C}_u$  in the generic fiber  $\mathbb{C}_u = \pi^{-1}(u)$ . The fundamental group  $\pi_1(\mathbb{C}_u - S, u_0)$  is freely generated by  $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$ , where  $\Gamma_j$  and  $\Gamma_{j'}$  are loops in  $\mathbb{C}_u$  around  $j$  and  $j'$  respectively. Let us explain how to create such generators. Define the generators  $\Gamma_j$  and  $\Gamma_{j'}$  in two steps. First select a path  $\alpha_j$  from  $u_0$  to a point  $u_j$  close to  $j$  and  $j'$ . Next choose small counterclockwise circles  $\eta_j$  and  $\eta'_{j'}$  starting from  $u_j$  and circling  $j$  and  $j'$  respectively. Use these three paths to build the generators  $\Gamma_j = \alpha_j \eta_j \alpha_j^{-1}$  and  $\Gamma_{j'} = \alpha_{j'} \eta'_{j'} \alpha_{j'}^{-1}$ .

By the van Kampen Lemma [15], there is a surjection from  $\pi_1(\mathbb{C}_u - S, u_0)$  onto  $\Pi_1 = \pi_1(\mathbb{C}^2 - S, u_0)$ . The images of  $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$  generate  $\Pi_1$ . By abuse of notation we denote them also by  $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$ . By the van Kampen Theorem [15], each braid in the braid monodromy factorization of  $S$  induces a relation on  $\Pi_1$  through its natural action on  $\mathbb{C}_u - \{j, j'\}_{j=1}^6$  [5]. A presentation for

$$(1) \quad \tilde{\Pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, u_0)}{\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle}$$

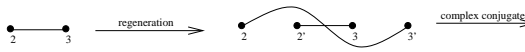
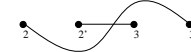
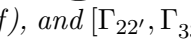
is thus immediately obtained from a presentation of  $\Pi_1$ .


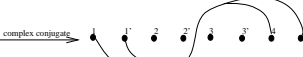
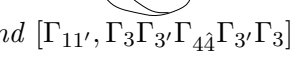
The algorithm used to compute a relation from a braid is explained in [7, Section 0.7], see also [1, Section 1.11].




Moishezon claimed in [5] that the braid monodromy factorization is invariant under complex conjugation of  $\mathbb{C}_u$ . Later it was proven in Lemma 19 of [10]. Therefore we can include the complex conjugate paths and relations in the table. For simplicity of notation we will use the following shorthand:  $\Gamma_{ii'}$  will stand for either  $\Gamma_i$  or  $\Gamma_{i'}$ ;  $\Gamma_{i'i}$  will stand for either  $\Gamma_{i'}\Gamma_i\Gamma_{i'}$  or  $\Gamma_{i'}$ ;  $\Gamma_{j\hat{j}}$  will stand for either  $\Gamma_j$  or  $\Gamma_j\Gamma_{j'}\Gamma_j$ .

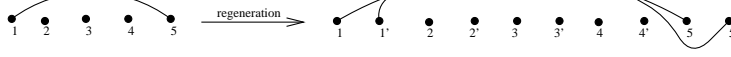
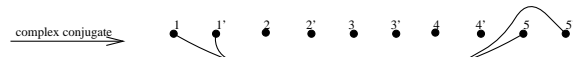



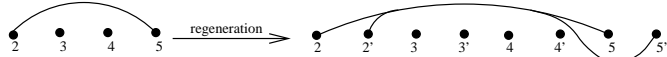

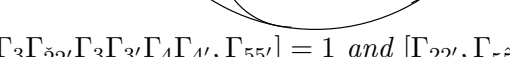
**Table 3.2.** We present the relations induced by the van Kampen Theorem from the paths, one for every pair of non-intersecting lines  $\hat{L}_i, \hat{L}_j$ .

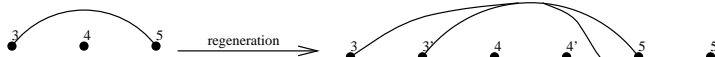
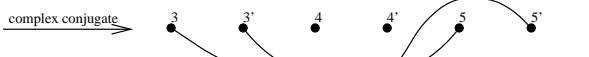
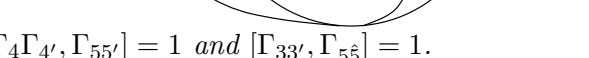
- $\tilde{Z}_{22',33'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

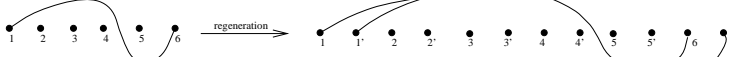
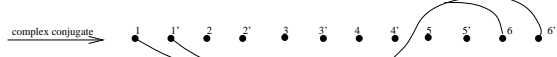
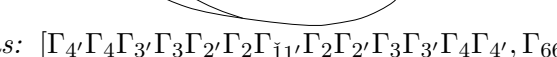
The relations:  $[\Gamma_{\tilde{2}2'}, \Gamma_{33'}] = 1$  (from the path itself), and  $[\Gamma_{22'}, \Gamma_{33}] = 1$  (from the complex conjugate).
- $\tilde{Z}_{11',44'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

Relations:  $[\Gamma_{2'}\Gamma_2\Gamma_{11'}\Gamma_2\Gamma_{2'}, \Gamma_{44'}] = 1$  and  $[\Gamma_{11'}, \Gamma_3\Gamma_3'\Gamma_{44'}\Gamma_3'\Gamma_3] = 1$ .
- $\tilde{Z}_{22',44'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

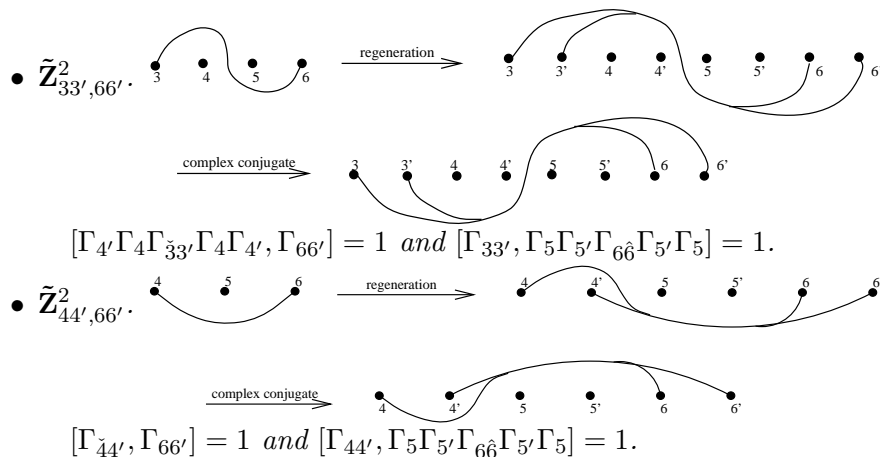
$[\Gamma_{\tilde{2}2'}, \Gamma_{44'}] = 1$  and  $[\Gamma_{22'}, \Gamma_3\Gamma_3'\Gamma_{44'}\Gamma_3'\Gamma_3] = 1$ .
- $\tilde{Z}_{11',55'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

$[\Gamma_{4'}\Gamma_4\Gamma_3'\Gamma_3\Gamma_2'\Gamma_2\Gamma_{11'}\Gamma_2\Gamma_2'\Gamma_3\Gamma_3'\Gamma_4\Gamma_{4'}, \Gamma_{55'}] = 1$  and  $[\Gamma_{11'}, \Gamma_{55}] = 1$ .
- $\tilde{Z}_{22',55'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

$[\Gamma_{4'}\Gamma_4\Gamma_3'\Gamma_3\Gamma_{\tilde{2}2'}\Gamma_3\Gamma_3'\Gamma_4\Gamma_{4'}, \Gamma_{55'}] = 1$  and  $[\Gamma_{22'}, \Gamma_{55}] = 1$ .
- $\tilde{Z}_{33',55'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

$[\Gamma_{4'}\Gamma_4\Gamma_{\tilde{3}3'}\Gamma_4\Gamma_{4'}, \Gamma_{55'}] = 1$  and  $[\Gamma_{33'}, \Gamma_{55}] = 1$ .
- $\tilde{Z}_{11',66'}^2$ .   $\xrightarrow{\text{regeneration}}$    $\xrightarrow{\text{complex conjugate}}$  

Relations:  $[\Gamma_{4'}\Gamma_4\Gamma_3'\Gamma_3\Gamma_2'\Gamma_2\Gamma_{11'}\Gamma_2\Gamma_2'\Gamma_3\Gamma_3'\Gamma_4\Gamma_{4'}, \Gamma_{66'}] = 1$  and  $[\Gamma_{11'}, \Gamma_5\Gamma_5'\Gamma_{66'}\Gamma_5'\Gamma_5] = 1$ .



**3.4. Checking Degrees.** Having computed the  $C'_i$  (Subsection 3.3) and the  $H_{V_i}$  (Figure 10), we obtain a regenerated braid monodromy factorization  $\Delta_S^2 = \prod_{i=1}^6 C'_i H_{V_i}$ . To verify that no factors are missing we compare degrees. First, since  $S$  is a curve of degree 12 (double the 6 lines in  $S_0$ ), the braid  $\Delta_S^2$  has degree  $12 \cdot 11 = 132$ . The six monodromies  $H_{V_i}$  each consist of three cusps and one branch point for a combined degree of  $6 \cdot (3 \cdot 3 + 1) = 60$ . The  $C'_i$  consist of four nodes for each parasitic intersection. The nine parasitic intersections (Table 3.2) give a combined degree of  $9 \cdot (4 \cdot 2) = 72$ . So together  $\prod C'_i H_{V_i}$  has also a degree of  $60 + 72 = 132$ , which proves that no factor was left out.

#### 4. INVARIANCE THEOREMS AND $\tilde{\Pi}_1$

**4.1. The Invariance Theorem.** Invariance properties are results concerning the behavior of a braid monodromy factorization under conjugation by certain elements of the braid group. A factorization  $g = g_1 \cdots g_k$  is said to be invariant under  $h$  if  $g = g_1 \cdots g_k$  is Hurwitz equivalent to  $(g_1)_h \cdots (g_k)_h$ . Geometrically this means that if a braid monodromy factorization of  $\Delta_C^2$  coming from a curve  $S$  is invariant under  $h$ , then the conjugate factorization is also a valid braid monodromy factorization for  $S$ .

The following rules [10, Section 3] give invariance properties of commonly occurring subsets of braid monodromy factorizations. Factors of the third type do not appear in our factorization.

- (a)  $\mathbf{Z}_{ii',jj'}^2$  is invariant under  $\mathbf{Z}_{ii'}^p$  and  $\mathbf{Z}_{jj'}^p$ ,  $\forall p \in \mathbb{Z}$ .
- (b)  $\mathbf{Z}_{i,jj'}^3$  is invariant under  $\mathbf{Z}_{jj'}^p$ ,  $\forall p \in \mathbb{Z}$ .
- (c)  $\mathbf{Z}_{ij}^1$  is invariant under  $\mathbf{Z}_{ii'}^p \mathbf{Z}_{jj'}^p$ ,  $\forall p \in \mathbb{Z}$ .

**Remark 4.1.** *The elements  $\mathbf{Z}_{ii'}$  and  $\mathbf{Z}_{jj'}$  commute for all  $1 \leq i, j \leq 6$  since the path from  $i$  to  $i'$  does not intersect the path from  $j$  to  $j'$ .*

**Theorem 4.2** (Invariance Theorem). *The braid monodromy factorization  $\Delta_{12}^2 = \prod_{i=1}^6 C'_i H_{V_i}$  is invariant under  $\prod_{j=1}^6 \mathbf{Z}_{jj'}^{m_j}$ , for all  $m_j \in \mathbb{Z}$ .*

*Proof.* It is sufficient to show that the  $C'_i$  and the  $H_{V_i}$  are invariant individually. Corollary 14 of [10] proves that each  $H_{V_i}$  is invariant under  $\mathbf{Z}_{jj'}$ ,  $1 \leq j \leq 6$ . Since the  $\mathbf{Z}_{jj'}$  all commute the invariance extends to arbitrary products  $\prod_{j=1}^6 \mathbf{Z}_{jj'}^{m_j}$ . The  $C'_i$  are composed of quadruples of factors  $\tilde{\mathbf{Z}}_{kk',\ell\ell'}$ , one from each parasitic intersection. Lemma 16 of that paper shows that each  $\tilde{\mathbf{Z}}_{kk',\ell\ell'}$  is invariant under  $\mathbf{Z}_{jj'}$ ,  $1 \leq j \leq 6$ . As before the invariance extends to products  $\prod_{j=1}^6 \mathbf{Z}_{jj'}^{m_j}$ . So the factorization  $\Delta_{12}^2 = \prod_{i=1}^6 C'_i H_{V_i}$  is invariant under conjugation by these elements.  $\square$

We use  $\Gamma_{(j)}$  to denote any element of the set  $\{(\Gamma_j)_{\mathbf{Z}_{jj'}^m}\}_{m \in \mathbb{Z}}$ . These elements are odd length alternating products of  $\Gamma_j$  and  $\Gamma_{j'}$ . Thus  $\Gamma_{(j)}$  represents any element of  $\{(\Gamma_j \Gamma_{j'})^p \Gamma_j\}_{p \in \mathbb{Z}}$ . The original generators  $\Gamma_j$  and  $\Gamma_{j'}$  are easily seen to be members of this set for  $p = 0, -1$ .

As an immediate consequence of the Invariance Theorem, any relation satisfied by  $\Gamma_j$  is satisfied by any element of  $\Gamma_{(j)}$ . This infinitely expands our collection of known relations in  $\tilde{\Pi}_1$ , however all of the new relations are consequences of our original finite set of relations.  $\Delta_C^2$  is also invariant under complex conjugation [10, Lemma 19], so we can use the complex conjugates  $\bar{H}_{V_i}$  and  $\bar{C}'_i$  to derive additional relations. Once again these relations are already implied by the existing relations. On the other hand, many of the complex conjugate braids in Table 3.2 have simpler paths than their counterparts so they are a useful tool.

The paths corresponding to the  $H_{V_i}$  are already quite simple (see Figure 10) so nothing is gained there by using complex conjugates.

**4.2. A presentation for  $\tilde{\Pi}_1$ .** Let  $S$  be the regenerated branch curve and let  $\pi_1(\mathbb{C}^2 - S, u_0)$  be the fundamental group of its complement in  $\mathbb{C}^2$ . We know that this group is generated by the elements  $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$ . Recall (Equation (1)) that  $\tilde{\Pi}_1 = \pi_1(\mathbb{C}^2 - S, u_0) / \langle \Gamma_j^2, \Gamma_{j'}^2 \rangle$ .

We have listed the braids  $C_i$  (Table 3.2) and  $H_{V_i}$  (Figure 10). These are the only braids in the factorization of  $\Delta_C^2$ , as explained in Subsection 3.4.

To the path of each braid there correspond two elements of  $\pi_1(\mathbb{C}^2 - S, u_0)$ , as explained in Subsection 3.3. From these, the van Kampen Theorem [15] produce the defining relations of  $\pi_1(\mathbb{C}^2 - S, u_0)$ .

**Theorem 4.3.** *The group  $\tilde{\Pi}_1$  is generated by  $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^6$  with the following relations:*

- (2)  $\Gamma_j^2 = 1 \quad j = 1, \dots, 6$
- (3)  $\Gamma_{j'}^2 = 1 \quad j = 1, \dots, 6$
- (4)  $\Gamma_{3'} = \Gamma_1 \Gamma_{1'} \Gamma_3 \Gamma_{1'} \Gamma_1$
- (5)  $\Gamma_{5'} = \Gamma_4 \Gamma_{4'} \Gamma_5 \Gamma_{4'} \Gamma_4$
- (6)  $\Gamma_2 = \Gamma_{6'} \Gamma_6 \Gamma_{2'} \Gamma_6 \Gamma_{6'}$
- (7)  $\Gamma_{2'} = \Gamma_1 \Gamma_{1'} \Gamma_2 \Gamma_{1'} \Gamma_1$
- (8)  $\Gamma_3 = \Gamma_{4'} \Gamma_4 \Gamma_3 \Gamma_4 \Gamma_{4'}$
- (9)  $\Gamma_5 = \Gamma_{6'} \Gamma_6 \Gamma_{5'} \Gamma_6 \Gamma_{6'}$
- (10)  $[\Gamma_{ii'}, \Gamma_{jj'}] = 1 \quad \text{if the lines } i, j \text{ are disjoint in } X_0$
- (11)  $\Gamma_{(i)} \Gamma_{(j)} \Gamma_{(i)} = \Gamma_{(j)} \Gamma_{(i)} \Gamma_{(j)} \quad \text{if the lines } i, j \text{ intersect.}$

The enumeration of the lines is given in Figure 4.  $\Gamma_{ii'}$  represents either  $\Gamma_i$  or  $\Gamma_{i'}$ , and  $\Gamma_{(i)}$  stands for any odd length word in the infinite dihedral group  $\langle \Gamma_i, \Gamma_{i'} \rangle$ .

*Proof.* Relations (2)–(3) hold in  $\tilde{\Pi}_1$  by assumption. The other relations hold in  $\pi_1(\mathbb{C}^2 - S, u_0)$ . To see this, we now list the relations induced by the  $H_{V_i}$  (Table 3.1). Recall that each  $H_{V_i}$  is a product of regenerated braids induced from one branch point (the second path in each part of Figure 10), and three cusps (condensed in the first path of each part). Applying the van Kampen Theorem [15], we have two types of relations. The relations (4)–(9) are derived from the branch point braids, and the triple relations  $\Gamma_i \Gamma_j \Gamma_i = \Gamma_j \Gamma_i \Gamma_j$  from the three other braids. Using the Invariance Theorem 4.2 to expand the pathes, we get (11) in its full generality. It remains to prove Equation (10). Note that the relations  $[\Gamma_{ii'}, \Gamma_{jj'}] = 1$  and  $[\Gamma_{(i)}, \Gamma_{(j)}] = 1$  imply each other.

We now consider the complex conjugates in Table 3.2. The relations  $[\Gamma_{(2)}, \Gamma_{(3)}]$ ,  $[\Gamma_{(2)}, \Gamma_{(4)}]$ ,  $[\Gamma_{(1)}, \Gamma_{(5)}]$ ,  $[\Gamma_{(2)}, \Gamma_{(5)}]$ ,  $[\Gamma_{(3)}, \Gamma_{(5)}]$ , and  $[\Gamma_{(4)}, \Gamma_{(6)}]$  appear rather directly. We must derive the other commutators, namely  $[\Gamma_{(1)}, \Gamma_{(4)}]$ ,  $[\Gamma_{(1)}, \Gamma_{(6)}]$ , and  $[\Gamma_{(3)}, \Gamma_{(6)}]$ .

By the second part of Table 3.2,

$$[\Gamma_{2'} \Gamma_2 \Gamma_{(1)} \Gamma_2 \Gamma_{2'}, \Gamma_{(4)}] = \Gamma_{2'} \Gamma_2 \Gamma_{(1)} \Gamma_2 \Gamma_{2'} \Gamma_{(4)} \Gamma_{2'} \Gamma_2 \Gamma_{(1)} \Gamma_2 \Gamma_{2'} \Gamma_{(4)},$$

but since  $[\Gamma_{(2)}, \Gamma_{(4)}] = 1$  we get

$$\Gamma_{2'} \Gamma_2 \Gamma_{(1)} \Gamma_{(4)} \Gamma_2 \Gamma_{2'} \Gamma_{2'} \Gamma_2 \Gamma_{(1)} \Gamma_{(4)} \Gamma_2 \Gamma_{2'} = \Gamma_{2'} \Gamma_2 \Gamma_{(1)} \Gamma_{(4)} \Gamma_{(1)} \Gamma_{(4)} \Gamma_2 \Gamma_{2'}$$

from which  $[\Gamma_{(1)}, \Gamma_{(4)}] = 1$  follows.

Now, by the seventh part of the table,

$$[\Gamma_{(1)}, \Gamma_5 \Gamma_5' \Gamma_{(6)} \Gamma_5' \Gamma_5] = \Gamma_{(1)} \Gamma_5 \Gamma_5' \Gamma_{(6)} \Gamma_5' \Gamma_5 \Gamma_{(1)} \Gamma_5 \Gamma_5' \Gamma_{(6)} \Gamma_5' \Gamma_5$$

but since  $[\Gamma_{(1)}, \Gamma_{(5)}] = 1$  we get

$$\Gamma_5 \Gamma_{5'} \Gamma_{(1)} \Gamma_{(6)} \Gamma_{5'} \Gamma_5 \Gamma_5 \Gamma_{5'} \Gamma_{(1)} \Gamma_{(6)} \Gamma_{5'} \Gamma_5 = \Gamma_5 \Gamma_{5'} \Gamma_{(1)} \Gamma_{(6)} \Gamma_{(1)} \Gamma_{(6)} \Gamma_5 \Gamma_{5'}$$

from which we get  $[\Gamma_{(1)}, \Gamma_{(6)}] = 1$ .

In the same way, since  $\Gamma_{(3)}$  and  $\Gamma_{(5)}$  commute we can get  $[\Gamma_{(3)}, \Gamma_{(6)}] = 1$  from the relation  $[\Gamma_{(3)}, \Gamma_5 \Gamma_{5'} \Gamma_{(6)} \Gamma_{5'} \Gamma_5]$  (the eighth part of the table). This finishes the proof of (10).  $\square$

### 5. THE HOMOMORPHISM $\psi$

$X^{\text{Aff}} - f^{-1}(S)$  is a degree 6 covering space of  $\mathbb{C}^2 - S$ . Let

$$\psi: \pi_1(\mathbb{C}^2 - S, u_0) \rightarrow S_6$$

be the permutation monodromy of this cover. As before let  $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}$  be a generic projection and choose  $u \in \mathbb{C}$  such that  $S$  is unramified over  $u$ . For surfaces  $X$  close to the degenerated  $X_0$  the points  $i$  and  $i'$  will be close to each other in  $\mathbb{C}_u$ . Finally choose a point  $u_0 \in \mathbb{C}_u$ .

We wish to determine what happens to the six preimages  $f^{-1}(u_0)$  in  $X$  as they follow the lifts of  $\Gamma_i$  and  $\Gamma_{i'}$ . Again, for surfaces  $X$  close to the degenerate  $X_0$  these six points inherit a unique enumeration based on which numbered plane of  $X_0$  they are nearest. This enumeration remains valid all along  $\alpha_i$  so we need only consider the monodromy around  $\eta_i$  and  $\eta'_i$ . Take a small neighborhood  $U_i \subset \mathbb{C}_u$  of  $i$  and  $i'$ . We can reduce the dimension of the question by restricting to  $f^{-1}(U_i)$  which is a branched cover of  $U_i$ , branched over  $i$  and  $i'$ . It is clear that  $f_0^{-1}(U_i)$  has a simple node over  $i$  where two planes containing  $\hat{L}_i$  meet. When  $i$  divides, the node will factor into two simple branch points over  $i$  and  $i'$  involving the same sheets which met at the node. Thus we see that if  $P_k$  and  $P_\ell$  intersect in  $\hat{L}_i$  then  $\psi(\Gamma_i) = \psi(\Gamma_{i'}) = (k \ell)$ . Specifically,  $\psi$  is defined by

**Definition 5.1.** *The map  $\psi: \pi_1(\mathbb{C}^2 - S, u_0) \rightarrow S_6$  is given by*

$$\psi(\Gamma_1) = \psi(\Gamma_{1'}) = (13),$$

$$\psi(\Gamma_2) = \psi(\Gamma_{2'}) = (15),$$

$$\psi(\Gamma_3) = \psi(\Gamma_{3'}) = (23),$$

$$\psi(\Gamma_4) = \psi(\Gamma_{4'}) = (26),$$

$$\psi(\Gamma_5) = \psi(\Gamma_{5'}) = (46),$$

$$\psi(\Gamma_6) = \psi(\Gamma_{6'}) = (45).$$

Figure 12 depicts the simplicial complex of  $X_0$  with the planes and intersection lines numbered. From this we can determine the values of  $\psi$  on the generators  $\Gamma_i$  and  $\Gamma_{i'}$ . Figure 13 below gives another graphical representation for  $\psi$ , in which  $\Gamma_i$  connects the two vertices  $\alpha, \beta$  defined by  $\psi(\Gamma_i) = (\alpha\beta)$ .

The reader may wish to check that  $\psi$  is well defined (testing the relations given in Theorem 4.3), but this is of course guaranteed by the theory. From the definition  $\psi$  is clearly surjective.

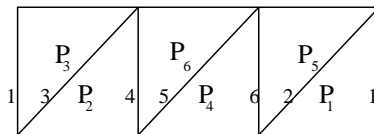


FIGURE 12

Since  $\psi(\Gamma_j^2) = 1$  and  $\psi(\Gamma_{j'}^2) = 1$ ,  $\psi$  also defines a map  $\tilde{\Pi}_1 \rightarrow S_6$ . We will use  $\psi$  to denote this map as well. Let  $\mathcal{A}$  be the kernel of  $\psi: \tilde{\Pi}_1 \rightarrow S_6$ . We have a short exact sequence

$$(12) \quad 1 \longrightarrow \mathcal{A} \longrightarrow \tilde{\Pi}_1 \xrightarrow{\psi} S_6 \rightarrow 1.$$

**Theorem 5.2** ([7]). *The fundamental group  $\pi_1(X_{\text{Gal}}^{\text{Aff}})$  is isomorphic to  $\mathcal{A}$ , where  $X_{\text{Gal}}^{\text{Aff}}$  is the affine part of  $X_{\text{Gal}}$  the Galois cover of  $X$  with respect to a generic projection onto  $\mathbb{CP}^2$ .*

## 6. A COXETER SUBGROUP OF $\tilde{\Pi}_1$

Our next step in identifying the group  $\tilde{\Pi}_1$  is to study a natural subgroup, which happens to be a Coxeter group.

Let  $\theta: \tilde{\Pi}_1 \rightarrow C$  be the map defined by  $\theta(\Gamma_j) = \theta(\Gamma_{j'}) = \gamma_j$ . The resulting group  $C = \text{Im}(\theta)$  is formally defined by the generators  $\gamma_1, \dots, \gamma_6$  and the relations obtained by applying  $\theta$  to the relations of  $\tilde{\Pi}_1$ .

Since we have  $\psi(\Gamma_j) = \psi(\Gamma_{j'})$ ,  $\psi$  splits through  $\theta$ : defining  $\rho: C \rightarrow S_6$  by  $\rho(\gamma_j) = \psi(\Gamma_j)$ , we have that  $\psi = \rho \circ \theta$ .

**Lemma 6.1.** *In terms of the generators  $\gamma_j$ ,  $C$  has the following presentation*

$$\begin{aligned} C = \langle \gamma_1, \dots, \gamma_6 \mid & \gamma_i^2 = 1, \\ & \gamma_1\gamma_3\gamma_1 = \gamma_3\gamma_1\gamma_3, \gamma_3\gamma_4\gamma_3 = \gamma_4\gamma_3\gamma_4, \gamma_4\gamma_5\gamma_4 = \gamma_5\gamma_4\gamma_5, \\ & \gamma_5\gamma_6\gamma_5 = \gamma_6\gamma_5\gamma_6, \gamma_6\gamma_2\gamma_6 = \gamma_2\gamma_6\gamma_2, \gamma_2\gamma_1\gamma_2 = \gamma_1\gamma_2\gamma_1, \\ & [\gamma_1, \gamma_4], [\gamma_1, \gamma_5], [\gamma_1, \gamma_6], [\gamma_3, \gamma_5], [\gamma_3, \gamma_6], [\gamma_3, \gamma_2], [\gamma_4, \gamma_6], [\gamma_4, \gamma_2], [\gamma_5, \gamma_2] \rangle. \end{aligned}$$

*Proof.* We only need to apply  $\theta$  on the relations of  $\tilde{\Pi}_1$  given in Theorem 4.3. Each of the relations in  $\tilde{\Pi}_1$  descends to a relation in  $C$ . The branch points all give relations of the form  $\Gamma_{3'} = \Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1$ . When we equate  $\Gamma_j = \Gamma_{j'}$  these relations vanish. If  $L_i, L_j$  intersect, then the relations  $\Gamma_{(i)}\Gamma_{(j)}\Gamma_{(i)} = \Gamma_{(j)}\Gamma_{(i)}\Gamma_{(j)}$  descend to  $\gamma_i\gamma_j\gamma_i = \gamma_j\gamma_i\gamma_j$ . Similarly if the lines  $\hat{L}_i, \hat{L}_j$  are disjoint, we get  $[\gamma_i, \gamma_j]$ .  $\square$

This presentation is easier to remember using Figure 13:  $\gamma_i, \gamma_j$  satisfy the triple relation if the corresponding lines intersect in a common vertex, and commute otherwise.

We will use Reidemeister-Schreier method to compute  $\mathcal{A}$ . For this we need to split  $\theta$ .

**Lemma 6.2.**  *$\theta$  is split by the map  $s: C \rightarrow \tilde{\Pi}_1$  defined by  $\gamma_j \mapsto \Gamma_j$ .*

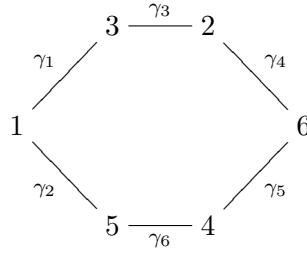


FIGURE 13

*Proof.* From the definition of  $\theta$  it is clear that  $\theta \circ s$  is the identity on  $C$ , so it remains to check that  $s$  is well defined. In Lemma 6.1 we gave a presentation of  $C$ . To prove that  $\gamma_j \mapsto \Gamma_j$  is a splitting we must show that  $\{\Gamma_j\}$  also satisfies the relations in  $\tilde{\Pi}_1$ .  $s$  respects  $\gamma_j^2 = 1$ , since  $\Gamma_j^2 = 1$  in  $\tilde{\Pi}_1$ . Let  $i, j$  be indices such that  $L_i, L_j$  intersect, then  $\gamma_i \gamma_j \gamma_i = \gamma_j \gamma_i \gamma_j$  is respected since in  $\tilde{\Pi}_1$  we have  $\Gamma_{(i)} \Gamma_{(j)} \Gamma_{(i)} = \Gamma_{(j)} \Gamma_{(i)} \Gamma_{(j)}$  so specifically  $\Gamma_i \Gamma_j \Gamma_i = \Gamma_j \Gamma_i \Gamma_j$ . Finally if  $i, j$  are such that  $\hat{L}_i \cap \hat{L}_j = \phi$ ,  $[\gamma_i, \gamma_j]$  is respected since  $[\Gamma_{ii'}, \Gamma_{jj'}] = 1$  for disjoint  $i, j$ .  $\square$

Observe that Lemma 6.1 presents  $C$  as a Coxeter group on the generators  $\gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_2$ , with a hexagon (the dual of that shown in Figure 13) as the Coxeter-Dynkin diagram of the group. The fundamental group of this defining graph is of course  $\mathbb{Z}$ . In previous works on the fundamental groups of Galois covers, the group  $C$  defined in a similar manner to what we define here, always happen to be equal to the symmetric group  $S_n$  (where  $n$  is the number of planes in the degeneration). Here, the map from  $C$  to  $S_6$  is certainly not injective ( $C$  is known [3] to be the group  $S_6 \rtimes \mathbb{Z}^5$ , with an action of  $S_6$  on  $\mathbb{Z}^5$  by the nontrivial component of the standard representation). The connection of this fact to the fundamental group of  $S_0$  is explained in more details in [2].

It will be useful for us to have a concrete isomorphism of  $C$  and  $S_6 \rtimes \mathbb{Z}^5$ .

**Lemma 6.3.**  $C \cong S_6 \rtimes \mathbb{Z}^5$  where  $\mathbb{Z}^5$  is the nontrivial component of the standard representation.

*Proof.* First note that  $\langle \gamma_2, \dots, \gamma_6 \rangle$  is the parabolic subgroup of  $C$  corresponding to the Dynkin diagram of type  $A_5$ , so that  $C_0 = \langle \gamma_2, \dots, \gamma_6 \rangle = S_6$ . We will therefore identify the subgroup  $C_0$  with the symmetric group  $S_6$  (using  $\psi$  as the identifying map). Next, note that  $\psi(\gamma_1) = (13)$ , so we set  $x = (13)\gamma_1$ , and consider the presentation of  $C$  on the new set of generators, namely  $x, \gamma_2, \dots, \gamma_6$ . Substituting  $\gamma_1 = (13)x$  in the presentation of Lemma

6.1, we obtain  $C = \langle x, S_6 \rangle$ , with the relations

$$\begin{aligned} (13)x(23)(13)x &= (23)(13)x(23), \\ (13)x(15)(13)x &= (15)(13)x(15), \\ (26)x(26) &= x, \\ (46)x(46) &= x, \\ (45)x(45) &= x. \end{aligned}$$

Define  $x_\sigma = \sigma x \sigma^{-1}$ , then the fact that  $x$  commute with  $\langle (26), (46), (45) \rangle = S_{\{2,4,5,6\}}$  shows that  $x_\sigma$  actually depends only on  $\sigma^{-1}(1), \sigma^{-1}(3)$ . We can thus define  $x_{k\ell} = x_\sigma$  for some  $\sigma \in \langle \gamma_2, \dots, \gamma_6 \rangle$  such that  $\sigma(k) = 1$  and  $\sigma(\ell) = 3$  (so in particular  $x_{13} = x$ ). With this definition one checks that  $\sigma^{-1}x_{k\ell}\sigma = x_{\sigma(k),\sigma(\ell)}$ . Adding this last relation as a definition of the  $x_{k\ell}$ , we obtain the following presentation:  $C = \langle x_{k\ell}, S_6 \rangle$ , with the relations

$$\begin{aligned} (13)x_{13}(23)(13)x_{13} &= (23)(13)x_{13}(23), \\ (13)x_{13}(15)(13)x_{13} &= (15)(13)x_{13}(15), \\ \sigma^{-1}x_{k\ell}\sigma &= x_{\sigma(k),\sigma(\ell)}. \end{aligned}$$

Now, the first two relations translate to

$$\begin{aligned} x_{32}x_{13} &= x_{12}, \\ x_{51}x_{13} &= x_{53}, \end{aligned}$$

which after conjugating by an arbitrary  $\sigma$  give

$$\begin{aligned} x_{ij}x_{ki} &= x_{kj}, \\ x_{ki}x_{ij} &= x_{kj}, \end{aligned}$$

which shows that  $\langle x_{k\ell} \rangle$  is generated by  $x_{12}, x_{13}, \dots, x_{16}$  and is commutative (using the fact that the  $x_{1i}$  commute). Thus  $\langle x_{k\ell} \rangle = \mathbb{Z}^5$ , and  $C = \langle \mathbb{Z}^5, S_6 \rangle$  is the asserted group.  $\square$

The inclusion  $S_6 \hookrightarrow C$  defined by sending the transpositions (15), (23), (26), (46) and (45) to  $\gamma_2, \dots, \gamma_6$  respectively, splits the projection  $\rho: C \rightarrow S_6$ . From now on we identify  $S_6$  with the subgroup  $\langle \gamma_2, \dots, \gamma_6 \rangle$  of  $C$ , as well as the subgroup  $\langle \Gamma_2, \dots, \Gamma_6 \rangle$  of  $\tilde{\Pi}_1$ .

**Corollary 6.4.** *The sequence (12) is split (by the composition of the maps  $S_6 \hookrightarrow C$  and  $s: C \hookrightarrow \tilde{\Pi}_1$ ). We denote the splitting map by  $\varphi$ .*

## 7. THE KERNEL OF $\psi$

We use the Reidemeister-Schreier method to find a presentation for the kernel  $\mathcal{A}$  of the map  $\psi: \tilde{\Pi}_1 \rightarrow S_6$ . Let  $L = \text{Ker}(\theta: \tilde{\Pi}_1 \rightarrow C)$  and  $K = \text{Ker}(\rho \circ \iota: C \rightarrow S_6)$ , and consider the diagram of Figure 14, in which the rows are exact by definition of  $\mathcal{A}$  and  $K$ , and the middle column by definition of  $L$ . The equality  $L = L$  in the diagram follows from the nine lemma. Then, since  $1 \rightarrow L \rightarrow \tilde{\Pi}_1 \rightarrow C \rightarrow 1$  splits (Lemma 6.2), we have that  $\mathcal{A}$  is



$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & L & \xlongequal{\quad} & L & & \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mathcal{A} & \longrightarrow & \tilde{\Pi}_1 & \xrightleftharpoons[\varphi]{\psi} & S_6 \longrightarrow 1 \\
 & & \updownarrow & & \updownarrow & & \theta \\
 1 & \longrightarrow & K & \longrightarrow & C & \xrightleftharpoons[\rho]{} & S_6 \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 
 \end{array}$$

FIGURE 14

a semidirect product of  $L = \text{Ker}(\theta: \tilde{\Pi}_1 \rightarrow C)$  and  $K = \text{Ker}(\rho: C \rightarrow S_6)$ , which is isomorphic to  $\mathbb{Z}^5$  by Lemma 6.3.

**7.1. The Reidemeister-Schreier method.** Let

$$1 \rightarrow K \rightarrow G \xrightarrow{\phi} H \rightarrow 1$$

be a short exact sequence, split by  $\rho: H \rightarrow G$ . Assume that  $G$  is finitely generated, with generators  $a_1, \dots, a_n$ . Then  $\rho\phi(g)$  is a representative for  $g \in G$  in its class modulo  $H$ . It is easy to see that  $K$  is generated by the elements  $ga_i(\rho\phi(ga_i))^{-1}$ ,  $1 \leq i \leq n$ ,  $g \in \rho(H)$ . Now,  $ga_i(\rho\phi(ga_i))^{-1} = ga_i(\rho\phi(a_i))^{-1}(\rho\phi(g))^{-1} = ga_i(\rho\phi(a_i))^{-1}g^{-1}$ , because  $\phi\rho$  is the identity on  $H$ . For  $g \in \rho(H)$ , denote the generators above by

$$\gamma(g, a_i) = ga_i(\rho\phi(a_i))^{-1}g^{-1}.$$

The relations of  $G$  can be translated into expressions in these generators by the following process. If the word  $\omega = a_{i_1} \cdots a_{i_t}$  represents an element of  $K$  then  $\omega$  can be rewritten as the product

$$\tau(\omega) = \gamma(1, a_{i_1})\gamma(\rho\phi(a_{i_1}), a_2) \cdots \gamma(\rho\phi(a_{i_1} \cdots a_{i_{t-1}}), a_{i_t}).$$

**Theorem 7.1** (Reidemeister-Schreier). *Let  $\{R\}$  be a complete set of relations for  $G$ . Then  $K = \text{Ker}(\phi)$  is generated by the  $\gamma(g, a_i)$  ( $1 \leq i \leq n$ ,  $g \in H$ ), with the relations  $\{\tau(trt^{-1})\}_{r \in R, t \in \rho(H)}$ .*

We will use this method to investigate  $L = \text{ker } \theta$  and  $\mathcal{A} = \text{ker } \psi$ .

**7.2. Generators for  $L = \text{Ker } \theta$ .** For  $c \in \text{Im}(s) = \langle \Gamma_1, \dots, \Gamma_6 \rangle$ , we let

$$(13) \quad A_{c,j} = c\Gamma_j\Gamma_j'c^{-1}.$$

We start with the following:

**Corollary 7.2.** *The group  $L = \text{Ker}(\theta)$  is generated by  $\{A_{c,j}\}_{1 \leq j \leq 6, c \in C}$ .*

*Proof.* By Theorem 7.1,  $L$  is generated by elements of the form  $c\Gamma_j(s\theta(\Gamma_j))^{-1}c^{-1}$  and  $c\Gamma_{j'}(s\theta(\Gamma_{j'}^{-1}))c^{-1}$  for all  $1 \leq j \leq 6$  and  $c \in C$ . We compute  $s\theta(\Gamma_j) = s(\gamma_j) = \Gamma_j$  and  $s\theta(\Gamma_{j'}) = s(\gamma_j) = \Gamma_j$ , so the generators are  $c\Gamma_j\Gamma_j^{-1}c^{-1} = I$  and  $c\Gamma_{j'}\Gamma_j^{-1}c^{-1} = c\Gamma_{j'}\Gamma_jc^{-1}$  using  $\Gamma_j^2 = I$ .  $\square$

This set of generators is highly redundant as we shall later see, but for now we turn our attention to  $\mathcal{A} = \text{Ker } \psi$ .

**7.3. Generators for  $\mathcal{A} = \text{Ker } \psi$ .** By Theorem 7.1,  $\mathcal{A}$  is generated by the elements  $\sigma\Gamma_j(\varphi\psi(\Gamma_j))^{-1}\sigma^{-1}$  and  $\sigma\Gamma_{j'}(\varphi\psi(\Gamma_{j'}))^{-1}\sigma^{-1}$ ,  $1 \leq j \leq 6$ ,  $\sigma \in S_6$ . Again we compute  $\varphi\psi(\Gamma_j)$  and  $\varphi\psi(\Gamma_{j'})$ . Recall that  $\langle \Gamma_2, \dots, \Gamma_6 \rangle = S_6$  is the image of  $\varphi$  (Corollary 6.4), so for  $j \neq 1$  we get  $\varphi\psi(\Gamma_j) = \varphi\psi(\Gamma_{j'}) = \Gamma_j$ , and the generators are

$$(14) \quad A_{\sigma,j} = \sigma\Gamma_j\Gamma_{j'}\sigma^{-1}.$$

This agrees with our previous definition of  $A_{c,j}$  for  $c \in S_6$ , see Equation (13).

The permutation (13) can be expressed in terms of the generators of  $S_6$  corresponding to  $\Gamma_2, \dots, \Gamma_6$  as follows:

$$(13) = (15)(54)(46)(26)(23)(26)(46)(54)(15),$$

so for  $j = 1$  we have that

$$\varphi\psi(\Gamma_1) = \varphi((15)(54)(46)(62)(23) \cdots (15)) = \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2.$$

Likewise,  $\varphi\psi(\Gamma_1) = \varphi\psi(\Gamma_{1'})$  since  $\psi(\Gamma_j) = \psi(\Gamma_{j'})$ . So we get generators

$$(15) \quad X_\sigma = \sigma\Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_1\sigma^{-1}$$

$$(16) \quad B_\sigma = \sigma\Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_{1'}\sigma^{-1}.$$

Since  $X_\sigma^{-1}B_\sigma = \sigma\Gamma_1\Gamma_{1'}\sigma^{-1} = A_{\sigma,1}$ , we have the following result:

**Corollary 7.3.** *The group  $\mathcal{A} = \text{Ker}(\psi)$  is generated by  $A_{\sigma,j}$ ,  $X_\sigma$ , for  $\sigma \in S_6$  and  $j = 1, \dots, 6$ .*

Notice that we are now conjugating only by permutations  $\sigma \in S_6$  instead of all elements  $c \in C$  (as in Corollary 7.2) so this is a finite set of generators.

## 8. A BETTER SET OF GENERATORS FOR $\mathcal{A}$

We first show that  $A_{\sigma,j}$  are not needed for  $j = 2, \dots, 6$

**Theorem 8.1.**  *$\mathcal{A}$  is generated by  $\{A_{\sigma,1}, X_\sigma\}$ .*

*Proof.* This follows immediately from the relations proven below.  $\square$

**Table 8.2.** *We have the following relations:*

$$\begin{aligned}
(17) \quad & A_{\sigma,3} = A_{\sigma(23),1}A_{\sigma,1}^{-1} \\
(18) \quad & A_{\sigma,3} = A_{\sigma(26)(23),4} \\
(19) \quad & A_{\sigma,5} = A_{\sigma(26)(46),4} \\
(20) \quad & A_{\sigma,5} = A_{\sigma(45)(46),6} \\
(21) \quad & A_{\sigma,2} = A_{\sigma(45)(15),6} \\
(22) \quad & A_{\sigma,2} = A_{\sigma(15),1}A_{\sigma,1}^{-1}.
\end{aligned}$$

*Proof.* We use the relations of Theorem 4.3. Let  $I$  denote the identity element of  $S_6$ , so that by definition  $A_{I,j} = \Gamma_j\Gamma_{j'}$ . From (4) we have  $1 = \Gamma_{3'}\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1 = (\Gamma_{3'}\Gamma_3)(\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3)(\Gamma_{1'}\Gamma_1) = A_{I,3}^{-1}A_{(23),1}A_{I,1}^{-1}$  so we get  $A_{I,3} = A_{(23),1}A_{I,1}^{-1}$ .

From (5) we have  $1 = \Gamma_{5'}\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{4'}\Gamma_4 = (\Gamma_{5'}\Gamma_5)\Gamma_5\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{4'}\Gamma_4 = (\Gamma_{5'}\Gamma_5)\Gamma_5\Gamma_4\Gamma_5\Gamma_{4'}\Gamma_5\Gamma_4 = (\Gamma_{5'}\Gamma_5)\Gamma_4\Gamma_5\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_4 = A_{I,5}^{-1}A_{(26)(46),4}$ .

From (6) we have  $1 = \Gamma_2\Gamma_{6'}\Gamma_6\Gamma_{2'}\Gamma_6\Gamma_{6'} = (\Gamma_2\Gamma_{6'}\Gamma_6\Gamma_2)(\Gamma_2\Gamma_{2'})(\Gamma_6\Gamma_{6'})$  so we get  $A_{I,2} = \Gamma_2\Gamma_{2'} = \Gamma_2\Gamma_6\Gamma_{6'}\Gamma_2\Gamma_{6'}\Gamma_6 = \Gamma_2\Gamma_6\Gamma_2\Gamma_{6'}\Gamma_2\Gamma_6 = \Gamma_6\Gamma_2\Gamma_6\Gamma_{6'}\Gamma_2\Gamma_6 = A_{(45)(15),6}$ .

From (7) we have  $1 = \Gamma_{2'}\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_1 = (\Gamma_{2'}\Gamma_2)(\Gamma_2\Gamma_1\Gamma_{1'}\Gamma_2)(\Gamma_{1'}\Gamma_1) = A_{I,2}^{-1}A_{(15),1}A_{I,1}^{-1}$  so we get  $A_{I,2} = A_{(15),1}A_{I,1}^{-1}$ .

From (8) we have  $1 = \Gamma_3\Gamma_{4'}\Gamma_4\Gamma_3\Gamma_4\Gamma_{4'} = (\Gamma_3\Gamma_{4'}\Gamma_4\Gamma_3)(\Gamma_3\Gamma_3')(\Gamma_4\Gamma_{4'})$  so we get  $\Gamma_3\Gamma_{3'} = \Gamma_3\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_4\Gamma_4 = \Gamma_3\Gamma_4\Gamma_3\Gamma_{4'}\Gamma_3\Gamma_4 = \Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_3\Gamma_4 = A_{(26)(23),4}$ .

Finally from (9) we have  $1 = \Gamma_5\Gamma_{6'}\Gamma_6\Gamma_5\Gamma_6\Gamma_{6'} = (\Gamma_5\Gamma_{6'}\Gamma_6\Gamma_5)(\Gamma_5\Gamma_5')(\Gamma_6\Gamma_{6'})$  so we get  $\Gamma_5\Gamma_5' = \Gamma_5\Gamma_6\Gamma_{6'}\Gamma_5\Gamma_6\Gamma_6 = \Gamma_5\Gamma_6\Gamma_5\Gamma_{6'}\Gamma_5\Gamma_6 = \Gamma_6\Gamma_5\Gamma_6\Gamma_{6'}\Gamma_5\Gamma_6 = A_{(45)(46),6}$ .  $\square$

One may be tempted to use  $(\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3)(\Gamma_{1'}\Gamma_1) = \Gamma_1\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_1$  in a similar manner to the other cases, to rewrite  $A_{\sigma(23),1}A_{\sigma,1}^{-1}$  of Equation (17) as a single element of the form  $A_{\sigma,1}$ ; however note that in the definition (14) we require  $\sigma \in S_6 = \langle \Gamma_2, \dots, \Gamma_6 \rangle$ , so we do not have the equality  $\Gamma_1\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_1 = A_{(23)(13),1}$ . The same remark applies for Equation (22).

Iterating the relations of Table 8.2, we obtain a new relation for  $\langle A_{\sigma,1} \rangle$ :

$$\begin{aligned}
A_{\sigma(23),1}A_{\sigma,1}^{-1} &= A_{\sigma,3} \\
&= A_{\sigma(26)(23),4} \\
&= A_{\sigma(26)(23)(46)(26),5} \\
&= A_{\sigma(26)(23)(46)(26)(45)(46),6} \\
&= A_{\sigma(26)(23)(46)(26)(45)(46)(15)(45),2} \\
&= A_{\sigma(36)(24)(56)(14),2} \\
&= A_{\sigma(36)(24)(56)(14)(15),1}A_{\sigma(36)(24)(56)(14),1}^{-1}
\end{aligned}$$

which may be rewritten as

$$(23) \quad A_{\sigma(23),1}A_{\sigma,1}^{-1} = A_{\sigma(142563),1}A_{\sigma(142)(356),1}^{-1}$$

**Lemma 8.3.** *For every  $\sigma \in S_6$ ,  $X_\sigma$  depends only on  $\sigma^{-1}(1)$  and  $\sigma^{-1}(3)$ .*

*Proof.* Viewing  $S_6 \subset C$  as subgroups of  $\tilde{\Pi}_1$  (using the embedding  $s: C \rightarrow \tilde{\Pi}_1$ ), the elements  $X_\sigma$  belong to  $C = \langle \Gamma_1, \Gamma_2, \dots, \Gamma_6 \rangle$  by their definition (15).

Applying the isomorphism  $C \cong S_6 \rtimes \mathbb{Z}^5$  of Lemma 6.3, we see that

$$\begin{aligned} X_\sigma &= \sigma\Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_1\sigma^{-1} \\ &= s(\sigma(15)(54)(46)(62)(23)(62)(46)(54)(15)(13)x_{13}\sigma^{-1}) \\ &= s(\sigma(13)(13)x_{13}\sigma^{-1}) \\ &= s(\sigma x_{13}\sigma^{-1}) = s(x_{\sigma^{-1}(1)\sigma^{-1}(3)}). \end{aligned}$$

□

A similar result holds for  $\{A_{\sigma,1}\}$ .

**Lemma 8.4.** *For every  $\sigma \in S_6$ ,  $A_{\sigma,1}$  depends only on  $\sigma^{-1}(1)$  and  $\sigma^{-1}(3)$ .*

*Proof.* If  $\sigma_1^{-1}(1) = \sigma_2^{-1}(1)$  and  $\sigma_1^{-1}(3) = \sigma_2^{-1}(3)$ , then  $\tau = \sigma_1^{-1}\sigma_2$  stabilizes 1, 3, so  $\tau \in S_{\{2,6,4,5\}} = \langle \Gamma_4, \Gamma_5, \Gamma_6 \rangle$  which commute with both  $\Gamma_1$  and  $\Gamma_{1'}$ . By definition (14),  $A_{\sigma_2,1} = \sigma_2\Gamma_1\Gamma_{1'}\sigma_2^{-1} = \sigma_1\tau\Gamma_1\Gamma_{1'}\tau^{-1}\sigma_1^{-1} = \sigma_1\Gamma_1\Gamma_{1'}\sigma_1^{-1} = A_{\sigma_1,1}$ . □

We can thus define

**Definition 8.5.** *For  $k, \ell = 1, \dots, 6$ ,  $A_{k\ell}$  and  $X_{k\ell}$  are defined by*

$$(24) \quad X_{k\ell} = \sigma\Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2\Gamma_1\sigma^{-1}$$

$$(25) \quad A_{k\ell} = \sigma\Gamma_1\Gamma_{1'}\sigma^{-1}$$

where  $\sigma \in S_6 = \langle \Gamma_2, \dots, \Gamma_6 \rangle$  is any permutation such that  $\sigma(k) = 1$  and  $\sigma(\ell) = 3$ .

We need to know the action of  $S_6$  on these generators:

**Proposition 8.6.** *For every  $\sigma \in S_6 = \langle \Gamma_2, \dots, \Gamma_6 \rangle$  and  $k, \ell = 1, \dots, 6$ , we have that*

$$(26) \quad \sigma^{-1}A_{k\ell}\sigma = A_{\sigma(k),\sigma(\ell)}$$

$$(27) \quad \sigma^{-1}X_{k\ell}\sigma = X_{\sigma(k),\sigma(\ell)}$$

*Proof.* Let  $\tau \in S_6$  be such that  $\tau(k) = 1$  and  $\tau(\ell) = 3$ . Since  $A_{13} = \Gamma_1\Gamma_{1'}$  by Definition 8.5, we have  $A_{k\ell} = \tau A_{13}\tau^{-1}$  and  $\sigma^{-1}A_{k\ell}\sigma = \sigma^{-1}\tau A_{13}\tau^{-1}\sigma = A_{\tau^{-1}\sigma(1),\tau^{-1}\sigma(3)} = A_{\sigma(k),\sigma(\ell)}$ . The same proof works for the  $X_{k\ell}$ . □

Note that  $B_{k\ell} = X_{k\ell}A_{k\ell}$  can be defined in the same manner, and have the same  $S_6$ -action.

From Theorem 8.1 (with a little help from Lemma 8.3 and Lemma 8.4), we obtain

**Corollary 8.7.** *The group  $\mathcal{A}$  is generated by  $\{A_{k\ell}, X_{k\ell}\}_{1 \leq k, \ell \leq 6}$ .*

Since  $\theta(X_{k\ell}) = x_{k\ell}$ , we already proved

**Corollary 8.8.** *The elements  $\{\theta(X_{k\ell})\}$  generate  $K = \text{Ker}(\rho: C \rightarrow S_6)$ .*

In the new language, Equation (23) (for  $\sigma = I$ ) can be written as  $A_{12}A_{13}^{-1} = A_{36}A_{26}^{-1}$ , so conjugating we get

$$(28) \quad A_{ij}A_{ik}^{-1} = A_{k\ell}A_{j\ell}^{-1}$$

for any four distinct indices  $i, j, k, \ell$ . Using three consecutive applications of the relation (28) we can also allow  $i = \ell$ , and using just two applications we can get:

$$(29) \quad A_{ij}A_{ik}^{-1} = A_{\ell j}A_{\ell k}^{-1}$$

for any distinct indices  $i, j, k$  and  $\ell \neq j, k$ . In view of (28) and (29) and Table 8.2 we can write a translation table for the remaining generators  $A_{\sigma, j}$  for  $j \neq 1$ .

**Table 8.9.**  $A_{\sigma, j}$  in terms of  $A_{k\ell}$

$$(30) \quad A_{\sigma, 1} = \sigma A_{13} \sigma^{-1}$$

$$(31) \quad A_{\sigma, 2} = \sigma A_{x1} A_{x5}^{-1} \sigma^{-1} = \sigma A_{5x} A_{1x}^{-1} \sigma^{-1} \quad \text{where } x \neq 1, 5$$

$$(32) \quad A_{\sigma, 3} = \sigma A_{x2} A_{x3}^{-1} \sigma^{-1} = \sigma A_{3x} A_{2x}^{-1} \sigma^{-1} \quad \text{where } x \neq 2, 3$$

$$(33) \quad A_{\sigma, 4} = \sigma A_{x6} A_{x2}^{-1} \sigma^{-1} = \sigma A_{2x} A_{6x}^{-1} \sigma^{-1} \quad \text{where } x \neq 2, 6$$

$$(34) \quad A_{\sigma, 5} = \sigma A_{x4} A_{x6}^{-1} \sigma^{-1} = \sigma A_{6x} A_{4x}^{-1} \sigma^{-1} \quad \text{where } x \neq 4, 6$$

$$(35) \quad A_{\sigma, 6} = \sigma A_{x5} A_{x4}^{-1} \sigma^{-1} = \sigma A_{4x} A_{5x}^{-1} \sigma^{-1} \quad \text{where } x \neq 4, 5$$

The indices 1 and 5 which appear in the formula for  $A_{\sigma, 2}$  arise because  $\psi(\Gamma_2) = (15)$ . The conjugations by  $\sigma$  change the indices as in equation (26). Similar for  $A_{\sigma, 3} \cdots A_{\sigma, 6}$ .

We have reduced the generating set for  $\mathcal{A}$  to  $\{X_{k\ell}, A_{k\ell}\}_{k \neq \ell}$  and we know that the subgroup  $K = \langle X_{k\ell} \rangle \cong \mathbb{Z}^5$ . Now we use the Reidmeister-Schreier rewriting process to translate all of the relations of  $\tilde{\Pi}_1$ . From now on, we denote  $\bar{g} = \varphi\psi(g)$  for every  $g \in \tilde{\Pi}_1$ . Using the notation of Subsection 7.1,  $\gamma(\sigma, \Gamma_j) = I$  and  $\gamma(\sigma, \Gamma_{j'}) = A_{\sigma, j}^{-1}$  for  $j \neq 1$ . For  $j = 1$ ,  $\gamma(\sigma, \Gamma_1) = X_{\sigma}^{-1}$ , and  $\gamma(\sigma, \Gamma_{1'}) = B_{\sigma}^{-1}$ . We begin by translating some of the relations which involve  $\Gamma_1$  but not  $\Gamma_{1'}$ . These will yield the relations among the  $X_{k\ell}$  which we already know, but the exercise is useful nonetheless because other elements will satisfy identical sets of relations.

$\Gamma_1 \Gamma_1 \xrightarrow{\tau} \gamma(I, \Gamma_1) \gamma(\bar{\Gamma}_1, \Gamma_1) = X_I^{-1} X_{(13)}^{-1} = X_{13}^{-1} X_{31}^{-1}$ , so we deduce that  $X_{31} = X_{13}^{-1}$  and conjugating we get

$$(36) \quad X_{\ell k} = X_{k\ell}^{-1}.$$

The relations  $[\Gamma_1, \Gamma_4]$ ,  $[\Gamma_1, \Gamma_5]$ , and  $[\Gamma_1, \Gamma_6]$  in  $\tilde{\Pi}_1$  produce the same relations on  $\langle X_{k\ell} \rangle$ .

Now we translate the triple relations (11) for  $i, j$  adjacent. We start for example with  $\Gamma_1 \Gamma_2 \Gamma_1 \Gamma_2 \Gamma_1 \Gamma_2$ . Later we will continue with  $\Gamma_{1'} \Gamma_2 \Gamma_{1'} \Gamma_2 \Gamma_{1'} \Gamma_2$  and

finish with  $(\Gamma_{1'}\Gamma_1\Gamma_{1'})\Gamma_2(\Gamma_{1'}\Gamma_1\Gamma_{1'})\Gamma_2(\Gamma_{1'}\Gamma_1\Gamma_{1'})\Gamma_2$ . For these two indices, there is no need to use any more relations from  $\Gamma_{(1)}\Gamma_{(2)}\Gamma_{(1)} = \Gamma_{(2)}\Gamma_{(1)}\Gamma_{(2)}$ , since the Invariance Theorem 4.2 showed that all of these relations were consequences of the three above.

The relation  $\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1\Gamma_2$  translates through  $\tau$  to the expression

$$\gamma(I, \Gamma_1)\gamma(\overline{\Gamma_1}, \Gamma_2)\gamma(\overline{\Gamma_1\Gamma_2}, \Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1}, \Gamma_2)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2}, \Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2\Gamma_1}, \Gamma_2).$$

But since  $\gamma(\sigma, \Gamma_2) = I$  we get  $\gamma(I, \Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2}, \Gamma_1)\gamma(\overline{\Gamma_1\Gamma_2\Gamma_1\Gamma_2}, \Gamma_1)$ , and since  $\overline{\Gamma_1\Gamma_2} = (13)(15) = (135)$  we can further simplify to  $X_I^{-1}X_{(135)}^{-1}X_{(531)}^{-1} = X_{13}^{-1}X_{51}^{-1}X_{35}^{-1}$ . Thus  $X_{35}X_{51}X_{13} = 1$  and including all conjugates

$$(37) \quad X_{kl}X_{\ell m}X_{mk} = 1.$$

Similarly the relation  $\Gamma_1\Gamma_3\Gamma_1\Gamma_3\Gamma_1\Gamma_3$  translates through  $\tau$  to the expression

$$\gamma(I, \Gamma_1)\gamma(\overline{\Gamma_1}, \Gamma_3)\gamma(\overline{\Gamma_1\Gamma_3}, \Gamma_1)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1}, \Gamma_3)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1\Gamma_3}, \Gamma_1)\gamma(\overline{\Gamma_1\Gamma_3\Gamma_1\Gamma_3\Gamma_1}, \Gamma_3)$$

which equals  $X_I^{-1}X_{(123)}^{-1}X_{(321)}^{-1} = X_{13}^{-1}X_{32}^{-1}X_{21}^{-1}$ . Thus  $X_{21}X_{32}X_{13} = 1$ , and conjugating we obtain

$$(38) \quad X_{\ell m}X_{kl}X_{mk} = 1.$$

Together the relations (36)–(38) show that  $\langle X_{\ell m} \rangle$  is generated by the five elements  $X_{12}, \dots, X_{16}$  which will commute, so that  $\langle X_{kl} \rangle \cong \mathbb{Z}^5$ . These are precisely the relations we expected among the  $X_{kl}$  and no more.

We continue with some of the relations of  $\tilde{\Pi}_1$  which involve  $\Gamma_{1'}$  but not  $\Gamma_1$ . These yield identical relations among the  $B_{kl}$ .

$\Gamma_{1'}\Gamma_{1'} \xrightarrow{\tau} \gamma(I, \Gamma_{1'})\gamma(\overline{\Gamma_{1'}}, \Gamma_{1'}) = B_I^{-1}B_{(13)}^{-1} = B_{13}^{-1}B_{31}^{-1}$ . Thus  $B_{31} = B_{13}^{-1}$  and by all conjugations  $B_{\ell k} = B_{k\ell}^{-1}$ .

The relations  $[\Gamma_{1'}, \Gamma_4]$ ,  $[\Gamma_{1'}, \Gamma_5]$ , and  $[\Gamma_{1'}, \Gamma_6]$  produce the same relations on  $\langle B_{\sigma} \rangle$ .

The relation  $\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2$  translates through  $\tau$  to the expression

$$\gamma(I, \Gamma_{1'})\gamma(\overline{\Gamma_{1'}}, \Gamma_2)\gamma(\overline{\Gamma_{1'}\Gamma_2}, \Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}}, \Gamma_2)\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2}, \Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2\Gamma_{1'}}, \Gamma_2).$$

But since  $\gamma(\sigma, \Gamma_2) = I$  we get  $\gamma(I, \Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2}, \Gamma_{1'})\gamma(\overline{\Gamma_{1'}\Gamma_2\Gamma_{1'}\Gamma_2}, \Gamma_{1'})$ , and since  $\overline{\Gamma_{1'}\Gamma_2} = (13)(15) = (135)$  we further simplify to  $B_I^{-1}B_{(135)}^{-1}B_{(531)}^{-1} = B_{13}^{-1}B_{51}^{-1}B_{35}^{-1}$ . Thus  $B_{35}B_{51}B_{13} = 1$  and including all conjugations we have  $B_{kl}B_{\ell m}B_{mk} = 1$ . Similarly the relation  $\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_3$  translates through  $\tau$  to the expression  $B_{13}^{-1}B_{32}^{-1}B_{21}^{-1}$ . Thus  $B_{21}B_{32}B_{13} = 1$  and including all conjugations  $B_{\ell m}B_{kl}B_{mk} = 1$ . By the arguments applied above for the  $X_{kl}$ , we also have that  $\langle B_{kl} \rangle \cong \mathbb{Z}^5$  generated by  $B_{1k}$ .

We finish with the last necessary triple relations. Note that  $\overline{\Gamma_{1'}\Gamma_1\Gamma_{1'}} = \overline{\Gamma_1}$  and  $\tau(\Gamma_{1'}\Gamma_1\Gamma_{1'}) = B_I^{-1}X_{(13)}^{-1}B_I^{-1} = B_{13}^{-1}X_{31}^{-1}B_{13}^{-1} = B_{13}^{-1}X_{13}B_{13}^{-1}$ . So if we define  $C_{kl}$  to be  $B_{kl}X_{kl}^{-1}B_{kl} = X_{kl}A_{kl}^2$  then the additional relations are  $C_{\ell k} = C_{k\ell}^{-1}$ ,  $C_{kl}C_{\ell m}C_{mk} = 1$ , and  $C_{\ell m}C_{kl}C_{mk} = 1$ . By the arguments above, the  $\{C_{kl}\}$  generate another copy  $\mathbb{Z}^5 \subset \mathcal{A}$ . In fact for each exponent

$n$  the elements  $X_{kl}A_{kl}^n = B_{kl}X_{kl}^{-1} \cdots X_{kl}^{-1}B_{kl}$  or  $A_{lk}^n = X_{kl}B_{kl}^{-1} \cdots B_{kl}^{-1}X_{kl}$  generate a subgroup isomorphic to  $\mathbb{Z}^5$ .

The relations computed thus far turn out to be all of the relations in  $\pi_1(X_{\text{Gal}}^{\text{Aff}})$ . Once we show that the remaining relations translated from  $\tilde{\Pi}_1$  are consequences of the relations above we will have proven the following theorem:

**Theorem 8.10.** *The fundamental group  $\pi_1(X_{\text{Gal}}^{\text{Aff}})$  is generated by elements  $\{X_{ij}, A_{ij}\}$  with the relations*

$$(39) \quad X_{ji}A_{ji}^n = (X_{ij}A_{ij}^n)^{-1},$$

$$(40) \quad (X_{ij}A_{ij}^n)(X_{jk}A_{jk}^n)(X_{ki}A_{ki}^n) = 1,$$

$$(41) \quad (X_{jk}A_{jk}^n)(X_{ij}A_{ij}^n)(X_{ki}A_{ki}^n) = 1,$$

$$(42) \quad A_{ij}A_{ik}^{-1} = A_{kl}A_{j\ell}^{-1}$$

for every  $n \in \mathbb{Z}$ .

Before we show that the remaining relations are redundant we prove that the relations above imply that some of the  $A_{kl}$  commute. We shall frequently use the fact that  $X_{kl}X_{lm} = X_{lm}X_{kl} = X_{km}$  which is a consequence of (36)–(38).  $B_{kl}$  and  $C_{kl}$  satisfy this as well.

**Lemma 8.11.** *In  $\pi_1(X_{\text{Gal}}^{\text{Aff}})$  we have  $[A_{ij}, A_{ik}] = 1$  and  $[A_{ji}, A_{ki}] = 1$  for distinct  $i, j, k$ .*

*Proof.* Starting with  $C_{ki}C_{jk}C_{ij} = 1$  and use the definition of  $C_{ij}$  to rewrite it as  $(B_{ki}X_{ik}B_{ki})(B_{jk}X_{kj}B_{jk})(B_{ij}X_{ji}B_{ij}) = B_{ki}X_{ik}B_{ji}X_{kj}B_{ik}X_{ji}B_{ij} = (B_{ki}X_{ik})(B_{ji}X_{ij})(X_{ki}B_{ik})(X_{ji}B_{ij}) = A_{ik}^{-1}A_{ij}^{-1}A_{ik}A_{ij} = 1$ . Thus the commutator  $[A_{ij}, A_{ik}] = 1$ . The relation (28) can be used to show that  $[A_{ji}, A_{ki}] = 1$  as well.  $\square$

Now we treat the remainder of the relations in  $\tilde{\Pi}_1$ . For  $j \neq 1$  the relation  $\Gamma_j\Gamma_j$  translates immediately to the null relation. Next consider  $\Gamma_j\Gamma_{j'}$ .

$\Gamma_{3'}\Gamma_{3'} \xrightarrow{\tau} A_{I,3}^{-1}A_{(23),3}^{-1}$ . But taking the inverse and using Table 8.9 we get  $A_{(23),3}A_{I,3} = (A_{x3}A_{x2}^{-1})(A_{x2}A_{x3}^{-1})$  which cancels completely. Identical computations treat all other values of  $j$ .

$\Gamma_1\Gamma_{4'}\Gamma_1\Gamma_{4'} \xrightarrow{\tau} X_I^{-1}A_{(13),4}^{-1}X_{(13)(26)}^{-1}A_{(26),4}^{-1}$  and taking the inverse we get  $A_{(26),4}X_{(13)(26)}A_{(13),4}X_I$ . Using Table 8.9 we have  $A_{12}A_{16}^{-1}X_{31}A_{36}A_{32}^{-1}X_{13} = (X_{21}B_{12})(B_{61}X_{16})X_{31}(X_{63}B_{36})(B_{23}X_{32})X_{13} = X_{21}B_{62}B_{26}X_{12} = 1$ . Similar calculations show that  $[\Gamma_1, \Gamma_{5'}] = 1$  and  $[\Gamma_1, \Gamma_{6'}] = 1$  are also redundant.

$\Gamma_{1'}\Gamma_{4'}\Gamma_{1'}\Gamma_{4'} \xrightarrow{\tau} B_I^{-1}A_{(13),4}^{-1}B_{(13)(26)}^{-1}A_{(26),4}^{-1}$  and taking the inverse we get  $A_{(26),4}B_{(13)(26)}A_{(13),4}B_I$ . Using Table 8.9 we have  $A_{12}A_{16}^{-1}B_{31}A_{36}A_{32}^{-1}B_{13}$ . Now, by Lemma 8.11 we can commute  $A_{12}$  and  $A_{16}$  as well as  $A_{36}$  and  $A_{32}$  to get  $A_{16}^{-1}A_{12}B_{31}A_{32}^{-1}A_{36}B_{13} = (B_{61}X_{16})(X_{21}B_{12})B_{31}(B_{23}X_{32})(X_{63}B_{36})B_{13} = B_{61}X_{26}X_{62}B_{16} = 1$ . Again, similar calculations work for  $[\Gamma_{1'}, \Gamma_{5'}] = 1$  and  $[\Gamma_{1'}, \Gamma_{6'}] = 1$ .

For non-adjacent  $i, j \neq 1$  the relation  $[\Gamma_i, \Gamma_j]$  translates directly to the null relation. So next we treat  $[\Gamma_{i'}, \Gamma_j]$ .

$\Gamma_{2'}\Gamma_3\Gamma_{2'}\Gamma_3 \xrightarrow{\tau} A_{I,2}^{-1}A_{(15)(23),2}^{-1}$  and inverting we have  $A_{(15)(23),2}A_{I,3} = (A_{x5}A_{x1}^{-1})(A_{x1}A_{x5}^{-1}) = 1$ . The same happens for every other non-adjacent pair  $i, j \neq 1$ .

$\Gamma_{2'}\Gamma_{3'}\Gamma_{2'}\Gamma_{3'} \xrightarrow{\tau} A_{I,2}^{-1}A_{(15),3}^{-1}A_{(15)(23),2}^{-1}A_{(23),3}^{-1}$  and taking inverses again we get  $A_{(23),3}A_{(15)(23),2}A_{(15),3}A_{I,2} = (A_{x3}A_{x2}^{-1})(A_{1y}A_{5y}^{-1})(A_{z2}A_{z3}^{-1})(A_{5w}A_{1w}^{-1})$ . Substituting specific values  $x = 1, y = 2, z = 5,$  and  $w = 3$  we get  $A_{13}A_{12}^{-1}A_{12}A_{52}^{-1}A_{52}A_{53}^{-1}A_{53}A_{13}^{-1} = 1$ . Identical arguments work for every other non-adjacent  $i, j \neq 1$ .

All that remains are the triple relations for  $i, j \neq 1$ . As before we need only three such relations for each pair of indices. The relation  $\Gamma_i\Gamma_j\Gamma_i\Gamma_j\Gamma_i\Gamma_j$  translates trivially, so we begin with  $\Gamma_{i'}\Gamma_j\Gamma_{i'}\Gamma_j\Gamma_{i'}\Gamma_j$ .

$\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_3\Gamma_{4'}\Gamma_3 \xrightarrow{\tau} A_{I,4}^{-1}A_{(632),4}^{-1}A_{(236),4}^{-1}$  and taking the inverse we get  $A_{(236),4}A_{(632),4}A_{I,4} = (A_{6x}A_{3x}^{-1})(A_{3x}A_{2x}^{-1})(A_{2x}A_{6x}^{-1}) = 1$ .

Finally consider  $\Gamma_4\Gamma_{4'}\Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_4\Gamma_3\Gamma_4\Gamma_{4'}\Gamma_4\Gamma_3 \xrightarrow{\tau} A_{(26),4}^{-1}A_{(36),4}^{-1}A_{(23),4}^{-1}$  and  $A_{(23),4}A_{(36),4}A_{(26),4} = (A_{x6}A_{x3}^{-1})(A_{x3}A_{x2}^{-1})(A_{x2}A_{x6}^{-1}) = 1$ . So all of the relations in  $\tilde{\Pi}_1$  are included in Theorem 8.10.

## 9. THE PROJECTIVE RELATION

To complete the computation of  $\pi_1(X_{\text{Gal}})$  we need only to add the projective relation

$$\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{2'}\Gamma_3\Gamma_{3'}\Gamma_4\Gamma_{4'}\Gamma_5\Gamma_{5'}\Gamma_6\Gamma_{6'} = 1.$$

This relation translates in  $\mathcal{A}$  as the product  $P = A_{I,1}A_{I,2}A_{I,3}A_{I,4}A_{I,5}A_{I,6}$ . We must translate the  $A_{I,j}$  to the language of the  $A_{kl}$ , using Table 8.9:

$P$  translates to  $A_{13}(A_{21}A_{25}^{-1})(A_{31}A_{21}^{-1})(A_{21}A_{61}^{-1})(A_{61}A_{41}^{-1})(A_{41}A_{51}^{-1})$  which cancels to  $A_{13}A_{21}A_{25}^{-1}A_{31}A_{51}^{-1}$ . Using Equation (28), we get  $A_{13}A_{21}A_{25}^{-1}A_{25}A_{23}^{-1}$ . Thus the projective relation may be written as  $A_{13}A_{21}A_{23}^{-1} = 1$  or equivalently  $A_{23} = A_{13}A_{21}$ . Conjugating, this becomes

$$(43) \quad A_{ij} = A_{kj}A_{ik}.$$

Substituting back into (28), writing  $A_{ij} = A_{kj}A_{ik}$  and  $A_{kl} = A_{j\ell}A_{kj}$ , we obtain

$$(44) \quad A_{kj}A_{j\ell} = A_{j\ell}A_{kj}.$$

**Lemma 9.1.** *The subgroup  $\langle A_{kl} \rangle$  of  $\pi_1(X_{\text{Gal}})$  is commutative of rank of at most 5*

*Proof.* We will compute the centralizer of  $A_{ij}$  for fixed  $i, j$ . Let  $i, j, k, \ell$  be four distinct indices. We already know from Lemma 8.11 that  $A_{ij}$  commutes with  $A_{ik}$  and  $A_{\ell j}$ . By equation (44) it also commutes with  $A_{ki}$  and  $A_{j\ell}$ . Now equation (43) allows us to write  $A_{kl} = A_{i\ell}A_{\ell j}$ , both of which commute with  $A_{ij}$ , so  $\langle A_{kl} \rangle$  is commutative.



Now, since  $A_{jk}A_{ij} = A_{ik}$  and  $A_{ik}A_{ji} = A_{jk}$ , we have  $A_{jk}A_{ij}A_{ji} = A_{ik}A_{ji} = A_{jk}$ , so that  $A_{ji} = A_{ij}^{-1}$ , the group is generated by the  $A_{1k}$  ( $k = 2, \dots, 6$ ), and the rank is at most 5.  $\square$

We see that  $\pi_1(X_{\text{Gal}}) = \langle A_{ij}, X_{ij} \rangle$  with the two subgroups  $\langle A_{ij} \rangle, \langle X_{ij} \rangle$  isomorphic to  $\mathbb{Z}^5$ . The only question left is how these two subgroups interact.

**Lemma 9.2.** *In  $\pi_1(X_{\text{Gal}})$  the  $A_{ij}$  and  $X_{k\ell}$  commute.*

*Proof.* We need only consider the commutators of  $A_{13}$  and  $X_{ij}$  since all others are merely conjugates of these. First consider the commutator  $[X_{13}, A_{13}]$ . Since  $X_{13} = (13)\Gamma_1$  (choose  $\sigma = 1$  in (24) and note that as elements of  $S_6$ , we have  $(13) = \Gamma_2\Gamma_6\Gamma_5\Gamma_4\Gamma_3\Gamma_4\Gamma_5\Gamma_6\Gamma_2$ ) and  $A_{13} = \Gamma_1\Gamma_{1'}$  this becomes  $(13)\Gamma_1(\Gamma_1\Gamma_{1'})\Gamma_1(13)A_{13}^{-1} = (13)\Gamma_{1'}\Gamma_1(13)A_{13}^{-1} = A_{31}^{-1}A_{13}^{-1} = A_{13}A_{13}^{-1} = 1$ . So  $X_{13}$  and  $A_{13}$  commute.

Next consider the commutator  $X_{12}A_{13}X_{12}^{-1}A_{13}^{-1}$ . By definition we have that  $X_{12} = (23)X_{13}(23) = (23)(13)\Gamma_1(23) = (321)\Gamma_1\Gamma_3$ . Thus the commutator becomes  $(321)\Gamma_1\Gamma_3(\Gamma_1\Gamma_{1'})\Gamma_3\Gamma_1(123)A_{13}^{-1}$ . We use the triple relations  $\Gamma_{(1)}\Gamma_{(3)}\Gamma_{(1)} = \Gamma_{(3)}\Gamma_{(1)}\Gamma_{(3)}$  to get  $(321)\Gamma_3\Gamma_1\Gamma_3\Gamma_{1'}\Gamma_3\Gamma_1(123)A_{13}^{-1} = (321)\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3\Gamma_{1'}\Gamma_1(123)A_{13}^{-1} = (321)\Gamma_3\Gamma_1\Gamma_{1'}\Gamma_3(123)(321)\Gamma_{1'}\Gamma_1(123)A_{13}^{-1}$  which is equal to  $A_{(321)\Gamma_3,1}A_{(321),1}^{-1}A_{13}^{-1} = A_{(12),1}A_{(321),1}^{-1}A_{13}^{-1} = A_{23}A_{21}^{-1}A_{13}^{-1} = 1$ , thus proving that  $X_{12}$  and  $A_{13}$  commute.

Conjugating by  $(2j)$  we see that  $X_{1j}$  commutes with  $A_{13}$  and since  $X_{ij} = X_{1i}^{-1}X_{1j}$  we see that every  $X_{ij}$  commutes with  $A_{13}$ .  $\square$

**Theorem 9.3.** *The fundamental group  $\pi_1(X_{\text{Gal}}) \cong \mathbb{Z}^{10}$ .*

*Proof.*  $\pi_1(X_{\text{Gal}})$  is generated by  $A_{1j}$  and  $X_{1j}$  which all commute. Hence the group they generate is  $\mathbb{Z}^{10}$ .  $\square$

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*E-mail address:* amram@mi.uni-erlangen.de, goldberg@math.colostate.edu,  
meirav,teicher@macs.biu.ac.il, vishne@math.huji.ac.il