POLYNOMIAL IDENTITIES OF $M_2(G)$

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ABSTRACT. We describe an efficient way to use the S_n-module structure in the computation of the multilinear identities of degree n of a given algebra. The method was used to show that $M_2(G)$ (where G is the Grassmann algebra) has identities of degree 8, but of no smaller degree. Explicit identities of degree 8 are given. It was also checked that $\mathrm{PIdeg}(M_{2,1}(G)) \geq 9$ and that $M_3(F)$ has no identities of degree ≤ 8 apart from the consequences of the standard identity s_6.

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1. INTRODUCTION

Let F be a field of characteristic 0. The set of multilinear identities of an (associative) F-algebra is an ideal of the (associative) free algebra over F, closed under endomorphisms. Such an ideal is called a T-ideal.

Let G denote the Grassmann algebra over F. The verbally-prime T-ideals of the free associative algebra (*i.e.* those prime with respect to T-ideals) were classified by Kemer ([5],[6]), and are the ideals of identities of three families of algebras: matrices over F, matrices over G, and superalgebras of matrices over G.

A lot is known about the structure 'at infinity' of the T-ideals of these algebras, such as the codimension growth, the character multiplicities and so on. But a complete description of the T-ideals (*i.e.* a set of generators as a T-ideal) is known only for a field F, for the Grassmann algebra G [7], for $M_{1,1}(G)$ [10] and for $M_2(F)$ [11].

Any PI-algebra over a field of characteristic zero satisfies multilinear identities. The smallest degree of such an identity is called the PI-degree of the algebra. The PI-degree of $M_n(F)$ is known to be $2n$ by Amitsur-Levitzki theorem, and obviously $\mathrm{PIdeg}(G) = 3$. But the PI-degree of $M_n(G)$ $(n \geq 2)$ or the superalgebras $M_{n,m}(G)$ $(n+m \geq 2,$ $n, m \geq 1)$ is still not known.

In the next section we discuss the S_n-module structure of the ideal of multilinear identities from the point of view of substitutions. We show

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how to compute the identities of small degree, and give the following applications:

1. The PI-degree of $M_2(G)$ equals 8 (a lower bound $\mathrm{PIdeg}(M_2(G)) > 6$ was known before, [9]). Explicit identities are also given.

2. Any multilinear identity of $M_3(F)$ which does not follow from the Amitsur-Levitzki identity s_6, must have degree ≥ 9 (the case of degree 7 is covered by [8]).

3. $M_3(F)$ has central identities of degree 8 which are not consequences of the Drenskey-Kasparian central identity [3].

4. $\mathrm{PIdeg}(M_{2,1}(G)) \geq 9$.

Details are given in Sections 4–5.

Some special identity of the subspace $\left\{\begin{pmatrix} x & y \\ z & -x \end{pmatrix} : x, y, z \in G\right\}$ of zero trace matrices in $M_2(G)$ is given in Section 6.

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2. The S_n-Module of Identities

The applicability of the representation theory of the symmetric group to multilinear identities is well recognized. In this section we describe the basics of this connection from a practical point of view, emphasizing the role of the ideal of substitutions. Throughout, F is an algebraically closed field of characteristic 0, and A a PI-algebra over F.

Let V_n denote the F-vector space of multilinear polynomials in the (non-commuting) variables $X_1, \dots, X_n$, and $F[S_n]$ the group algebra of the symmetric group on n letters. We multiply permutations by the rule $(\pi\sigma)i = \pi(\sigma(i))$. The map $\sigma \mapsto X_{\sigma 1} \dots X_{\sigma n}$ is a natural isomorphism of vector spaces $F[S_n] \to V_n$, and the induced left action of S_n on V_n is given by

$$\pi \cdot X_{\sigma 1} \dots X_{\sigma n} = X_{\pi\sigma 1} \dots X_{\pi\sigma n}.$$

An element $f = \sum \alpha_\sigma X_{\sigma 1} \dots X_{\sigma n} \in V_n$ is an identity of A if for every substitution $X_i \mapsto x_i \in A$ we have that $f(x_1, \dots, x_n) = \sum \alpha_\sigma x_{\sigma 1} \dots x_{\sigma n} = 0$. Let $I = Id_n(A)$ denote the set of multilinear identities of degree n satisfied by the algebra A. Since $\pi \cdot f(X_1, \dots, X_n) = f(X_{\pi 1}, \dots, X_{\pi n})$, this set is closed under the above left action of S_n, so that I is a submodule of V_n. We identify I with its pre-image in $F[S_n]$, which is then a left ideal.

More generally, let $B \subseteq A$ be a linear subspace of A. I_B denotes the set of B-identities of A: multilinear expressions $f \in V_n$, such that $f(x_1, \dots, x_n) \in B$ for every substitution $X_i \mapsto x_i$. For example 0-identities are just identities, and $\mathrm{Cent}(A)$-identities are central identities (this term includes 0-identities).

For every additive functional $\mu : A \to F$ and specialization $X_i \mapsto x_i \in A$, define

$$u_\mu(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \mu(x_{\sigma 1} \dots x_{\sigma n})\sigma^{-1} \in F[S_n].$$

By $B^\perp$ we denote set of all linear functionals $\mu : A \to F$, such that $\mu(B) = 0$. Let $U_B = \{u_\mu(x_1, \dots, x_n) | x_i \in A, \mu \in B^\perp\} \subseteq F[S_n]$. Note that

$$u_\mu(x_{\pi 1}, \dots, x_{\pi n}) = u_\mu(x_1, \dots, x_n)\pi,$$

so that U_B is a right ideal of $F[S_n]$.

Let $\delta_\pi : F[S_n] \to F$ denote the map returning the coefficient of the element $\pi \in S_n$.

Remark 2.1. *$f \in F[S_n]$ is a B-identity of A iff $\delta_1(fu) = 0$ for every $u \in U_B$.*

Proof. Let $f = \sum_{\sigma \in S_n} f_\sigma \sigma$. Compute that

$$f \cdot u_\mu(x_1, \dots, x_n) = \sum_{\pi \in S_n} \left(\sum_{\sigma \in S_n} f_{\pi\sigma} \mu(x_{\sigma 1} \dots x_{\sigma n}) \right) \pi.$$

Now, by definition f is a B-identity iff $\sum_\sigma f_\sigma x_{\sigma 1} \dots x_{\sigma n} \in B$ for every $x_1, \dots, x_n$, which is the case iff for every functional $\mu : A \to F$ such that $\mu(B) = 0$, we have that $\delta_1(fu) = \mu(\sum_\sigma f_\sigma x_{\sigma 1} \dots x_{\sigma n}) = 0$. $\square$

Proposition 2.2. *The set I_B of B-identities is the left annihilator of U_B.*

Proof. Let $f \in F[S_n]$. If $fU_B = 0$ then f is a B-identity by the last remark. On the other hand, suppose $f \in I_B$ is a B-identity, and let $u \in U_B$. Since I_B is a left ideal, we have that $\delta_\pi(fu) = \delta_1(\pi^{-1} f \cdot u) = 0$ for every $\pi \in S_n$, and $fu = 0$. $\square$

For the rest of this section we set $B = 0$, so that $I = I_0$ is the left ideal of identities.

Let $\{\rho\}$ be the irreducible representations of S_n. Writing $F[S_n] = \oplus_\rho M_{\dim \rho}(F)$, we can decompose the left ideal I as a sum of left ideals of matrices, where every ρ-component is a row space in the respective matrices, the left annihilator of the ρ-component of U.

Example 2.3. *The standard identity $s_n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma$ generates an ideal of dimension 1 in $F[S_n]$. It covers the component of the sign representation, but is zero in any other representation.*

Note that every other identity generates a left ideal of dimension at least $n-1$ (since the trivial representation corresponds to $\sum \sigma$ which is not an identity in characteristic 0). In this sense s_n is the weakest among all identities of degree n.

We recall some basic facts regarding the Grassmann algebra. If V is an infinite dimensional vector space, $T(V) = F + V + V^{\otimes 2} + \dots$ denotes the tensor algebra of V. The Grassmann algebra of V is $G = T(V)/\langle v \otimes v | v \in V\rangle$. Note that every $v_1, v_2 \in V$ anti-commute, so that every monomial of even length is central. It is then easy to check that every commutator is in the center, so that G satisfies $[x,[y,z]] = 0$ and is PI. On the other hand it is known [7] that every identity of G is a consequence of this identity.

Example 2.4. *Consider the identity* $w = [x_1,[x_2,x_3]] = x_1x_2x_3 - x_1x_3x_2 - x_2x_3x_1 + x_3x_2x_1$ *of the Grassmann algebra* G. *The above described correspondence between monomials and permutations, sends* $w \mapsto (1) - (23) - (123) + (13) \in F[S_3]$.

Letting $\mathbf{1}, \mathrm{sgn}$ *denote the trivial and the sign representations of* $F[S_3]$, *it is easy to compute that* $\mathbf{1}(w) = \mathrm{sgn}(w) = 0$. *The standard representation of* S_3 *is given by the action on a basis* $\{e_1, e_2, e_3\}$ *by indices. Let* ϕ *denote the irreducible representation of dimension* 2 *defined by the action on* $\mathrm{span}\,\{e_1 - e_2, e_2 - e_3\}$. *Computing, we get* $\phi(w) = \begin{pmatrix} 0 & 0 \\ -3 & 3 \end{pmatrix}$; *the left ideal generated by* w *is thus seen to have dimension* 2.

3. Computation of Ideals of Identities

The basic strategy in finding the ideal I of identities of degree n of a given algebra is simple: write down the ideal U, and compute its left annihilator I. In order to prevent working with large subspaces of the $n!$-dimensional space $F[S_n]$, we decompose this algebra to its irreducible components, corresponding to the irreducible representations of S_n.

Computing with representations of S_n seems like a straightforward task, but it becomes very quickly impractical, even using a computer.

We now describe the method used to compute the identities of degree 8 for $M_2(G)$. Slight changes are needed when working with other matrix algebras, or when computing ideals of B-identities for $B \neq 0$.

The computation of ideals of identities for matrices over G is based on the following observation.

Remark 3.1. *Let* $V = \operatorname{span}\{v_1, \dots, v_n, \dots\}$ *be a generating linear space for the Grassmann algebra* G*. Let*

$$f(X_1, \dots, X_n) = \sum_{\sigma \in S_n} \alpha_\sigma X_{\sigma(1)} \dots X_{\sigma(n)}$$

be a multilinear expression over the base field. Then f *is an identity of the matrix algebra* $M_s(G)$ *iff, for every choice of matrix units* e_{a_i,b_i} *and either* $v_i^* = v_i$ *or* $v_i^* = 1$*, the substitution* $X_i \mapsto e_{a_i,b_i} v_i^*$ *in* f *gives zero.*

Proof. The algebra $M_s(G)$ is linearly spanned by elements of the form $e_{a,b}w$, where w is a monomial on the basis elements $\{v_j\}$. Thus, in order to check if f is an identity, it is enough to substitute elements of this form. Then, $f(X_1, \dots, X_n)$ is a matrix over the base field, multiplied by a monomial on $\{v_j\}$. However, since every product $v_j v_{j'}$ is central, the coefficient is not changed if we omit such products from the X_i, so that we may assume X_i is either a matrix unit, or a matrix unit times a basis element. Finally, reordering $\{v_j\}$, we may assume $X_i \mapsto e_{a_i,b_i} v_i^*$, as asserted. □

We are now ready to find the identities of $M_2(G)$.

Step 1. Prepare a list of all the directed graphs of n edges on 2 vertices, which have at least one Hamiltonian path (the graphs are listed up to isomorphism with respect to the starting and ending point of the pathes). An edge from i to j corresponds to a matrix unit e_{ij}, so the Hamiltonian paths correspond to non-zero products of matrix units. It speed things up if we sort the graphs so that those with smaller number of paths come first. This correspondence of pathes in graphs and products of matrix units was first exploited by R. Swan is his graph-theoretic proof of Amitsur-Levitzki theorem [14].

Step 2. For every graph, consider the substitution $X_i \mapsto e_{r_i s_i} v_i^*$ where $e_{r_1 s_1} \dots e_{r_n s_n}$ corresponds to a Hamiltonian path in the graph, r_1 is the starting point and s_n the ending point. We compute the set $Z \subseteq S_n$ of all $\sigma \in S_n$ such that $e_{\sigma s_1, \sigma r_1} \dots e_{\sigma s_n, \sigma r_n} = e_{s_1, r_n}$.

We now go over all the subsets $J \subseteq \{1, \dots, n\}$, and take $v_i^* = 1$ if $i \in J$, and $v_i^* = v_i$ otherwise. For every $\sigma \in Z$, we reorder every monomial $v_{\sigma 1}^* \dots v_{\sigma n}^*$ to the form $c_\sigma^{(J)} \cdot v_1^* \dots v_n^*$, with $c_\sigma^{(J)} = \pm 1$. The resulting elements $u^{(J)} = \sum_{\sigma \in Z} c_\sigma^{(J)} \sigma$ are in U. By Remark 3.1, these elements generate the ideal U. The $u^{(J)}$ have only $|Z|$ non-zero coefficients, so if this is a small number, it may happens that $u^{(J)} = u^{(J')}$ for some $J \neq J'$, and for efficiency we keep only one occurence of each element.

Step 3. For every representation ρ, compute the values $\rho(u^{(J)})$. Note that $\rho(u^{(J)}) = \rho(\sum_{\sigma\in Z} c_\sigma^{(J)}\sigma) = \sum_{\sigma\in Z} c_\sigma^{(J)}\rho(\sigma)$, so it is enough to compute $\rho(\sigma)$ only once for every $\sigma \in Z$. Then we add them according to the different signs (at most) 2^n times, and get $\rho(u^{(J)})$ for every J.

Step 4. Compute the column space generated by all the matrices $\rho(u^{(J)})$, together with those from previous graphs. If, at some point, the column space becomes the whole representation space, we don't need to check more graphs for this representation. Otherwise, go back to step 2.

Step 5. Finally, after $\rho(U)$ was computed for every ρ, we compute its left annihilator which is $\rho(I)$ by Proposition 2.2. An element $f \in I$ is determined by the images $\rho(f)$. Thus, if an orthogonal representation was used in the computations, we can recapture the coefficients of an explicit identity $f = \sum_\sigma \alpha_\sigma \sigma \in I$ by

$$\alpha_\sigma = \frac{1}{|S_n|}\sum_\rho \dim(\rho)\cdot \mathrm{tr}(\rho(f)\rho(\sigma^{-1})).$$

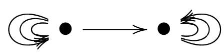
4. Multilinear Identities of $M_2(G)$

The method described in the previous section was used to compute the ideal of identities of degree $n = 8$ for the algebra $A = M_2(G)$.

Computing $\rho(U)$ for every irreducible representations ρ of S_7, we were able to check the following.

Proposition 4.1. *The PI-degree of $M_2(G)$ is at least* 8.

It is interesting to note that one graph was enough to show that $\rho(U)$ is the whole representation space for all representations:



This graph has only 36 Hamiltonian paths (this is the minimum for graphs with 7 edges on 2 vertices), so the representations were computed for relatively few elements.

Even more is true: there is no multilinear polynomials of degree 7 such that always $\mathrm{tr}(f(X_1,\dots,X_7)) = 0$, or such that $f(X_1,\dots,X_7)$ is always central.

In degree 8 we had to go through all the 37 different graphs. Conveniently, it was later found that here too one graph holds all the information:

The ideal generated by substitutions in this graph contains the ideals of all the other graphs. There are several other graphs with this property, but the one we choose has only 288 Hamiltonian paths, and is the most efficient (there are four graphs with 144 Hamiltonian paths, but their ideals are too small).

The ideal I of 8-degree multilinear identities of $M_2(G)$ has rank 18 (and dimension 880). Out of the 22 irreducible representations of S_8, there are 7 on which this ideal vanishes. It has rank 1 on 12 representations (corresponding to the partitions $8 \vdash (2,2,1,1,1,1), 8 \vdash (2,2,2,1,1), 8 \vdash (3,1,1,1,1,1), 8 \vdash (3,2,1,1,1), 8 \vdash (3,2,2,1), 8 \vdash (4,1,1,1,1), 8 \vdash (4,3,1), 8 \vdash (5,1,1,1), 8 \vdash (5,2,1), 8 \vdash (5,3), 8 \vdash (6,1,1)$ and $8 \vdash (6,2)$) and rank 2 on the remaining three ones ($8 \vdash (3,3,1,1), 8 \vdash (4,2,1,1)$ and $8 \vdash (4,2,2)$).

Each of these components was explicitly computed as a left ideal of the respective matrix algebra. Even if one believes that every component is generated by easy-to-describe identities (whatever that means), a random element of the ideal is a random sum of permuted versions of the generators, which can look very peculiar. Out of the 15 non-zero components of I we got lucky in four cases (all of rank 1), for which we present explicit identities.

To this end we need some notation. A *pattern* is a finite sequence of the letters A, B. If π is a pattern with a appearances of A and b of B, we denote by $\pi(x_1,\dots,x_a;y_1,\dots,y_b)$ the product of variables where the x's and y's are combined according to π. For example, $ABBA(x_1,x_2;y_1,y_2) = x_1y_1y_2x_2$. A coefficient in front of a pattern π means the monomial should be multiplied by that coefficient. Now let

$$P_\pi^+ = \sum_{\sigma\in S_a,\tau\in S_b} \operatorname{sgn}(\sigma)\pi(x_{\sigma 1},\dots x_{\sigma a};y_{\tau 1},\dots,y_{\tau b}),$$

(this is sometimes called a Capelli-type polynomial), and

$$P_\pi^- = \sum_{\sigma\in S_a,\tau\in S_b} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\pi(x_{\sigma 1},\dots x_{\sigma a};y_{\tau 1},\dots,y_{\tau b}).$$

Let

$$\mathcal{P}_1 = \left\{ \begin{array}{lll} +AAAABAAB, & +AABBAAAA, & -AABAAAAB, \\ -AAAABBAA, & -BAABAAAA, & +BAAAABAA \end{array} \right\}.$$

The component of I in the representation $8 \vdash (2,2,1,1,1,1)$ contains (and is thus generated by) $\sum_{\pi\in\mathcal{P}_1} P_\pi^-(x_1,\dots,x_6;y_1,y_2)$. Similarly, the component of I in the representation $8 \vdash (3,1,1,1,1,1)$ contains $\sum_{\pi\in\mathcal{P}_1} P_\pi^+(x_1,\dots,x_6;y_1,y_2)$. Note that in the sum

$$T_1(x_1,\dots,x_6;y_1,y_2) = \sum_{\pi\in\mathcal{P}_1} (P_\pi^- + P_\pi^+),$$

only the monomials with y_1 preceding y_2 appear; on the other hand $T_1(\dots;y_1,y_2) \pm T_1(\dots;y_2,y_1)$ gives back the original two identities.

The same phenomenon happens for another couple of representations. Let

$$\mathcal{P}_2 = \left\{ \begin{array}{lll} -AAABAABB, & -AABBAABA, & +ABBAABAA, \\ +AAABBAAB, & +AABAABBA, & -ABAABBAA, \\ -ABBAAAAB, & +BAABBAAA, & -BAAAABBA, \\ +ABAAAABB, & -BBAABAAA, & +BBAAAABA \end{array} \right\}.$$

The component in $8 \vdash (2,2,2,1,1)$ contains $\sum_{\pi\in\mathcal{P}_2} P^-_\pi(x_1,\dots,x_5;y_1,y_2,y_3)$, and the component in $8 \vdash (4,1,1,1,1)$ contains $\sum_{\pi\in\mathcal{P}_2} P^+_\pi(x_1,\dots,x_5;y_1,y_2,y_3)$. Again

$$T_2(x_1,\dots,x_5;y_1,y_2,y_3) = \sum_{\pi\in\mathcal{P}_2} (P^-_\pi + P^+_\pi)$$

has only the monomials in which the order of y_1, y_2, y_3 is even, and $T_2(\dots;y_1,y_2,y_3) \pm T_2(\dots;y_3,y_2,y_1)$ gives the original identities.

Corollary 4.2. *T_1, T_2 are multilinear identities of degree 8 of $M_2(G)$. In particular* $\mathrm{PIdeg} M_2(G) = 8$.

Let $B = \{a \in M_2(G) : \mathrm{tr}(a) = 0\}$ and $C = \mathrm{Cent}(M_2(G))$. The ideals of B-identities and C-identities of degree 8 of $M_2(G)$ were computed, and found to be identical to the ideal of identities.

Corollary 4.3. *Let f be a multilinear polynomial of degree 8. If*

$$\mathrm{tr} f(x_1,\dots,x_8) = 0$$

for every $x_1,\dots,x_8 \in M_2(G)$, then f is an identity of $M_2(G)$.

Corollary 4.4. *Every central identity of degree 8 of $M_2(G)$ is an identity.*

Finally, from the general result on the PI-degree of matrix algebras in [1], we get the following:

Corollary 4.5. *If n is even, then* $\mathrm{PIdeg} M_n(G) \geq 4n$.

5. Remarks on $M_3(F)$ and $M_{2,1}(G)$

The algebra $M_3(F)$ of matrices over a field is known to satisfy the standard identity s_6. In degree 7 the consequences are identities of the forms $x_1 s_6(x_2,\dots,x_7)$, $s_6(x_1,\dots,x_6)x_7$ and $s_6(x_1x_2,x_3,\dots,x_7)$. The dimension of the generated ideal is $7^2 - 1 = 48$ (see, *e.g.*, [8]). We directly computed the identities going over all representations of S_7, and got the same dimension.

The same computation was done for $n = 8$. Let $L_{\mathcal{I}}$ denote the homogeneous part of degree 8 of the T-ideal generated by s_6. As a left ideal of V_8, it is generated by the following elements: $x_1x_2s_6(x_3,\dots,x_8)$, $x_1s_6(x_2x_3,x_4,\dots,x_8)$, $x_1s_6(x_2,\dots,x_7)x_8$, $s_6(x_1x_2,x_3\dots,x_7)x_8$, $s_6(x_1,\dots,x_6)x_7x_8$, $s_6(x_1x_2,x_3x_4,x_5,\dots,x_8)$ and $s_6(x_1x_2x_3,x_4,\dots,x_8)$. The dimension of $L_{\mathcal{I}}$ was directly computed, and found to be 1469. On the other hand we computed the left ideal of identities of $M_3(F)$, and got precisely the same dimension. We conclude the following:

Theorem 5.1. *Any multilinear identity f of $M_3(F)$ which does not follow from s_6, has degree ≥ 9.*

Similar methods were used to study central identities of degree 8 for $M_3(F)$. Recall that $L_{\mathcal{I}}$ is the left ideal of identities. An explicit central identity (which is not an identity) was found by Drensky and Kasparian [3]; let $L_{\mathcal{DK}}$ denote the left ideal of $F[S_8]$ generated by this element. $L_{\mathcal{I}} + L_{\mathcal{DK}}$ are the central identities currently known.

Let $L_{\mathcal{C}}$ denote the left ideal of central identities of degree $n = 8$ of $M_3(F)$, so that obviously $L_{\mathcal{I}} + L_{\mathcal{DK}} \subseteq L_{\mathcal{C}}$.

For every irreducible representation ρ, the table below gives the rank of each of these ideals in the corresponding component of $F[S_8]$ (the representations for which $\rho(L_{\mathcal{C}}) = 0$ were omitted).

Young diagram	$\dim(\rho)$	$\rho(L_{\mathcal{DK}})$	$\rho(L_{\mathcal{I}})$	$\rho(L_{\mathcal{DK}} + L_{\mathcal{I}})$	$\rho(L_{\mathcal{C}})$
$(1,1,1,1,1,1,1,1)$	1	1	1	1	1
$(2,1,1,1,1,1,1)$	7	1	5	5	5
$(2,2,1,1,1,1)$	20	2	8	8	8
$(2,2,2,1,1)$	28	1	4	4	4
$(2,2,2,2)$	14	1	0	1	1
$(3,1,1,1,1,1)$	21	0	9	9	9
$(3,2,1,1,1)$	64	1	8	8	8
$(3,2,2,1)$	70	0	2	2	3
$(3,3,1,1)$	56	1	0	1	2
$(4,1,1,1,1)$	35	0	4	4	4
$(4,2,1,1)$	90	0	2	2	2
Total rank		8	43	45	47
Total dimension	8!	210	1469	1539	1665

It follows that $L_{\mathcal{I}} + L_{\mathcal{DK}} \subset L_{\mathcal{C}}$, so we conclude the following.

Corollary 5.2. *There exist central identities of degree 8 of $M_3(F)$ which are not a consequence of the Drensky-Kasparian central identity.*

It was also checked that the superalgebra $M_{2,1}(G)$ has no identites of degree 8, that is

Proposition 5.3. PIdeg$M_{2,1}(G) \geq 9$.

This may be contrasted with [12], where it was proved that $M_{k,\ell}$ has (super)trace-identities of degree $k\ell + k + \ell$.

6. Zero Trace Matrices in $M_2(G)$

For an algebra A and a multilinear polynomial $f(X_1, \dots, X_n)$, we denote by $f(A)$ the vector space spanned by all the substitutions $f(x_1, \dots, x_n)$, $x_i \in A$. This is a useful tool in finding identities of A, as it may happen that certain verbal subspaces have smaller PI-degree than that of A. If $A_1, \dots, A_n \subseteq A$, $f(A_1, \dots, A_n)$ is the vector space spanned by the substitutions $f(x_1, \dots, x_n)$ for $x_i \in A_i$.

Denote by s_n the standard polynomial $s_n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{\sigma 1} \dots X_{\sigma n}$, and similarly $a_n = \sum_{\sigma \in S_n} X_{\sigma 1} \dots X_{\sigma n}$. Note that $a_2 = X_1X_2 + X_2X_1$ is the multilinarization of $f(X) = X^2$. The composition $f \circ g$ denotes substitution of $\deg(f)$ *distinct* instances of g in f.

Let $G = G_0 + G_1$ be the decomposition of G into the even and odd parts, respectively. G_0^+ denotes the positive part of G_0. Let

$$S = \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix}, \qquad T = G\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $B = S + T$ denote the subspace of zero-trace matrices in $M_2(G)$.

Let $G_0^+ = G_{(2)} + G_{(4)} + \dots$ be the decomposition into homogeneous spaces, and $G_0^{++} = G_{(4)} + \dots$. It is easy to check that $s_2(G) = G_0^+$ and that $a_2(G) = G$. Direct computation shows that $a_2(S,S) = \begin{pmatrix} s_2(G) & 0 \\ 0 & s_2(G) \end{pmatrix} + a_2(G) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} G_0^+ & 0 \\ 0 & G_0^+ \end{pmatrix} + G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $a_2(S,T) = \begin{pmatrix} 0 & s_2(G) \\ s_2(G) & 0 \end{pmatrix} = \begin{pmatrix} 0 & G_0^+ \\ G_0^+ & 0 \end{pmatrix}$, and $a_2(T,T) = a_2(G) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = G \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Summing up, it follows that $a_2(B) = M_2(G_0^+) + G\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In a similar way one can check that $(s_2 \circ a_2)(B) = M_2(G_0^{++}) + G_{(2)} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \subseteq M_2(G_0^+)$ consists of matrices over the commutative ring G_0^+.

Corollary 6.1. *B satisfies the multilinear identity $s_4 \circ s_2 \circ a_2$ of degree 16.*

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