π_1-CLASSIFICATION OF REAL ARRANGEMENTS WITH UP TO EIGHT LINES

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ABSTRACT. One of the open questions in the geometry of line arrangements is to what extent does the incidence lattice of an arrangement determine its fundamental group. Line arrangements of up to 6 lines were recently classified by K.M. Fan [Fa2], and it turns out that the incidence lattice of such arrangements determines the projective fundamental group. We use actions on the set of wiring diagrams, introduced in [GTV], to classify real arrangements of up to 8 lines. In particular, we show that the incidence lattice of such arrangements determines both the affine and the projective fundamental groups.

Submitted, 7/2001.

1. INTRODUCTION

A *line arrangement* in $\mathbb{C}^2$ is a union of finitely many copies of $\mathbb{C}^1$. An arrangement is equipped with several invariants, probably the most important are the fundamental groups of the complement in $\mathbb{C}^2$ and in $\mathbb{CP}^2$. These are called the *affine* and the *projective fundamental group* of the arrangement, respectively.

A more combinatorial invariant is the incidence lattice of the arrangement. It is not known whether or not the lattice determines the fundamental groups [CS].

In 1997, Fan [Fa2] studied the projective fundamental group of arrangements with small number of lines. Using classification of the arrangements with 6 lines, he was able to show that if there are up to 6 lines, the incidence lattice determines the projective fundamental group.

An arrangement is called *real* if the defining equations of its lines can be written with real coefficients, and *complex* otherwise.

Rybnikov, in an unpublished work [Ry], presents two complex arrangements with 13 lines based on MacLane configuration, which have the same lattice, but different projective fundamental groups. This example was not thoroughly checked, and in any case the question for real arrangements remains open.

To a real arrangement, one can associate combinatorial objects: the wiring diagram and its associated list of Lefschetz pairs. They are not invariants, as they depend on the choice of a guiding generic line. Still, these objects turn out to be useful tools in the study of fundamental groups of line arrangements, *cf.* [MoTe], [GaTe], [CS].

The purpose of this paper is twofold. We extend Fan's result to arrangements of up to 8 lines (for the affine and projective fundamental groups), and at the same time present algorithms developed for this classification, which can be also used in other classification problems. While it is difficult to enumerate geometric objects as lines (the defining coefficients have infinitely many possible values), it is possible, in principle, to enumerate the wiring diagrams (or the lists of Lefschetz pairs) induced by them. However, many wiring diagrams are induced by the same line arrangement (because they depend on the guiding line), and we do not want to waste time studying various diagrams induced by the same arrangement.

To this end, we use equivalence relations and actions on the set of wiring diagrams which were introduced in [GTV]. We proved there that the equivalence relations and the actions preserve our two main invariants: the incidence lattice and the affine and projective fundamental groups. These results enable us to list all the incidence lattices of arrangements up to 8 lines. They also reduce the number of comparisons of pairs of fundamental groups needed to test the conjecture from over 20 millions to about 200 pairs. Eventually, we conclude that for real arrangements with up to 8 lines the incidence lattice determines both the affine and the projective fundamental groups.

The paper is organized as follows. In Section 2 we briefly recall some combinatorial objects related to a line arrangement: the wiring diagram and the associated list of Lefschetz pairs, the incidence lattice and the signature.

In section 3 we discuss the affine and projective fundamental groups of some special types of line arrangements. This is later used to avoid some cases in the classification.

In section 4, we briefly survey equivalence relations and actions on the set of wiring diagrams of a given signature. These were introduced and studied in [GTV].

In Sections 6–8 we tabulate all the real line arrangements of up to 8 lines according to their incidence lattices (some details on the arrangements of 6 lines were already given by Fan [Fa2]). The classification is achieved by classifying the appropriate lists of Lefschetz pairs, and requires considerable computer work. Details on the algorithms used are given in Section 5. Our main interest in this paper is the algorithms and the resulting classification. For more details (and proofs) on the actions and relations defined here, the reader is directed to [GTV].

2. Combinatorial objects related to line arrangements

We briefly recall some of the combinatorial objects and construction related to real line arrangements. For more details, see [GP] and [GTV].

2.1. Wiring diagrams and Lefschetz pairs.

Given a real line arrangement in $\mathbb{C}^2$, its intersection with the natural copy of $\mathbb{R}^2$ in $\mathbb{C}^2$ is an arrangement of lines in the real plane.

To an arrangement of ℓ lines in $\mathbb{R}^2$ one can associate a *wiring diagram* [Go], which holds the information on the intersection points and the relative position of their projection on a generic line. A wiring diagram is a collection of ℓ pseudolines (where a *pseudoline* in $\mathbb{R}^2$ is a union of segments and rays, homeomorphic to $\mathbb{R}$). A line L is *generic* with respect to a line arrangement $\mathcal{L}$, if it avoids all the intersection points of the arrangement, and the projections of intersection points on L do not overlap. Fixing a generic line L, the wiring diagram induced by $\mathcal{L}$ (with respect to L) contains ℓ pseudolines, and is constructed as follows: start at the '$-\infty$' side of L with ℓ parallel rays, and for every projection of an intersection point, make the corresponding switch in the rays.

To a wiring diagram, one can associate a list of *Lefschetz pairs*. Any pair of this list corresponds to one of the intersection points, and holds the first and the last indices of the pseudolines intersected at that point, numerated locally near the intersection point, from bottom to top. An example is given in Figure 1.

To every list of Lefschetz pairs there corresponds a wiring diagram, which is constructed by the reverse procedure. It should be noted that in the wiring diagram only the order of the intersection points is relevant, and not their distances.
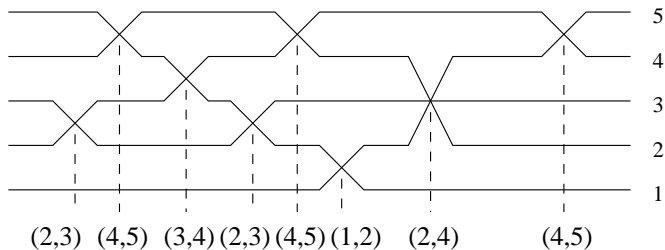
FIGURE 1

2.2. The incidence lattice.

Let $\mathcal{L} = \{L_1, L_2, \cdots, L_\ell\}$ be a line arrangement. By $\mathrm{Lat}(\mathcal{L})$ we denote the partially-ordered set of non-empty intersections of the L_i, ordered by inclusion (see [OrT]). We include the whole plane and the empty set in $\mathrm{Lat}(\mathcal{L})$, so that it becomes a lattice of height 3: the intersection points have height 1, and the lines height 2.

Equivalently, the lines and points form a bipartite graph, in which a line L_i and a point p_j are connected iff $p_j \in L_i$.

2.3. Lattice isomorphism and the fundamental group.

As mentioned in the introduction, it was conjectured by Cohen and Suciu [CS] that for real arrangements the incidence lattice determines the (affine and projective) fundamental groups. There is an unpulished example of Rybnikov [Ry] which shows that this is not the case for complex arrangements (at least for the projective fundamental group).

A rather weak version of that conjecture was recently proved by Cordovil [Co]. He introduces a new order (called *shellability order*) on the lines of an arrangement. Then, he shows that if there is a bijection of two arrangements which induces an isomorphism of the associated lattices, preserving the shellability order, then their affine fundamental groups are isomorphic.

2.4. The signature.

Another characteristic of a line arrangement is the so-called *signature* - the number of lines meeting in every intersection point.

Definition 2.1. *The* signature *of a line arrangement $\mathcal{L}$ is $[2^{n_2} 3^{n_3} \dots]$, where n_k is the number of points in which k lines intersect. We make the agreement to omit every c^0.*

For example, an arrangement which consists of five lines, four of which intersect at one point, has signature $[2^4 4^1]$. The signature is obviously determined by the incidence lattice.

The definition of the signature applies to any wiring diagram (and thus also to the associated list of Lefschetz pairs).

A point in which only two lines intersect is called *simple*, so that n_2 is the number of simple points in the arrangement. The other points are called *multiple*.

Note that different incidence lattices may still have the same signature. For example, both the following two arrangements have the same signature $[2^9 3^2]$, but their incidence lattices are not isomorphic (the two multiple points are connected in the first arrangement but not in the second):

(1)   (2)
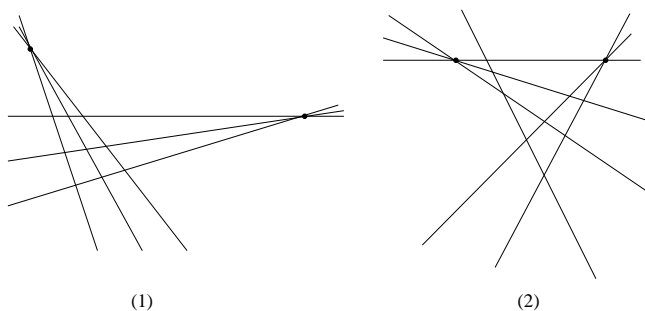
FIGURE 2

Remark 2.2. *The affine fundamental groups of both arrangements are isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{Z}^2$, and the projective fundamental groups are isomorphic to $\mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{Z}$, [GaTe]. Here $\mathbb{F}_2$ is the free group on two generators.*

It is not difficult to check that the example above is minimal: for less than 6 lines, the signature of an arrangement does determine the incidence lattice.

We say that a wiring diagram has the *unique intersection property*, if every two wires intersect exactly once. Obviously, a wiring diagram induced by a line arrangement has this property.

Lemma 2.3. *A wiring diagram has the unique intersection property if and only if the following conditions hold:*

   (1) *Its signature $S = [2^{n_2} 3^{n_3} \cdots]$ satisfies*

$$(1) \qquad \sum_{k \geq 2} n_k \binom{k}{2} = \binom{\ell}{2},$$

   where ℓ is the number of the wires in the diagram, and
   (2) *The order of the wires at $+\infty$ is the opposite of that at $-\infty$.*

Proof. The unique intersection property implies that the total number of intersection points is $\binom{\ell}{2}$. Moreover, if a wire x was below a wire y at $+\infty$ it will be above it from their intersection point on.

Conversely, if every two pseudolines changed order, then we have at least $\binom{\ell}{2}$ intersection points, and since this is the total number, every two lines intersect exactly once. $\square$

Note that not every signature satisfying Equation (1) can arise, an easy counterexample is given in the following remark:

Remark 2.4. *There is no wiring diagram (with $\ell = 4$ pseudolines) whose signature is $[3^2]$ (for if three lines intersect at one point, the remaining line must either join them at that point, or be a simple line, resulting in signature $[4^1]$ or $[2^3 3^1]$).*

3. The fundamental group of a line arrangement

In this section we discuss the fundamental groups of some special types of real line arrangements. This will be later used to analyse some special signatures.

3.1. **Arrangements with no cycles of multiple points.** The graph $G(\mathcal{L})$ of multiple points of a line arrangement $\mathcal{L}$ lies on the arrangement. It consists of the multiple points of $\mathcal{L}$, with the lines on which there are at least two multiple points. If two such lines happen to intersect in a simple point, it is ignored (and the lines are not considered to intersect in the graph theoretic sense).

Fan has proved the following result:

Theorem 3.1 ([Fa2, Theorem 3.2]). *If $G(\mathcal{L})$ has no cycles, then the projective fundamental group of $\mathcal{L}$ is*

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \mathbb{F}_{m_1-1} \oplus \cdots \oplus \mathbb{F}_{m_k-1} \oplus \mathbb{Z}^{\ell-(\sum_{i=1}^{k}(m_i-1))-1},$$

where $m_1, \cdots, m_k$ are the multiplicities of the multiple intersection points in $\mathcal{L}$ and ℓ is the number of lines.

A somewhat weaker result can be proved, using braid monodromy techniques and the van Kampen theorem, for the affine fundamental group:

Theorem 3.2 ([GaTe, Theorem 5.3]). *If every connected component of $G(\mathcal{L})$ consists of vertices lying on a straight line, then the affine fundamental group is:*

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \mathbb{F}_{m_1-1} \oplus \cdots \oplus \mathbb{F}_{m_k-1} \oplus \mathbb{Z}^{\ell-(\sum_{i=1}^{k}(m_i-1))}$$

where $m_1, \cdots, m_k$ and ℓ are as in Theorem 3.1.

It immediately follows that if $\mathcal{L}$ has at most two multiple points, then the (affine and projective) fundamental groups depend only on the signature of $\mathcal{L}$.

For other results on the topology of the complement of real arrangements, see [Ra], [Sa], [Ar], or [OrSo].

3.2. **Arrangements with simple lines.** We call a line $L \in \mathcal{L}$ *simple* if it has only simple intersection points. Note that if $\mathcal{L} = \mathcal{L}' \cup L$ where L is a simple line in $\mathcal{L}$, then the incidence lattice of $\mathcal{L}$ is determined by that of $\mathcal{L}'$.

The following result of Oka and Sakamoto allows us to reduce (for affine fundamental groups) to the case of no simple lines:

Proposition 3.3 ([OkSa]). *Let C_1 and C_2 be algebraic plane curves in $\mathbb{C}^2$ of degrees d_1, d_2 respectively. Assume that the intersection $C_1 \cap C_2$ consists of $d_1 d_2$ distinct points. Then:*

$$\pi_1(\mathbb{C}^2 - (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 - C_1) \oplus \pi_1(\mathbb{C}^2 - C_2).$$

In particular, if $\mathcal{L} = \mathcal{L}' \cup L$ where L is a simple line in $\mathcal{L}$, then:

$$\pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \pi_1(\mathbb{C}^2 - \mathcal{L}') \oplus \pi_1(\mathbb{C}^2 - L) \cong \pi_1(\mathbb{C}^2 - \mathcal{L}') \oplus \mathbb{Z}$$

Fan [Fa2] has proved a similar result for the projective fundamental group:

Proposition 3.4 ([Fa2, Lemma 3.1]). *Let $\mathcal{L}_1, \mathcal{L}_2 \subset \mathbb{CP}^2$ be arrangements of n_1, n_2 lines respectively, such that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at $n_1 n_2$ distinct points in $\mathbb{CP}^2$. Let L be a line which does not contain any of these intersection points, and let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup L \subset \mathbb{CP}^2$. Then:*

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - (\mathcal{L}_1 \cup L)) \oplus \pi_1(\mathbb{CP}^2 - (\mathcal{L}_2 \cup L)).$$

In particular, if $\mathcal{L} = \mathcal{L}' \cup L$ is a line arrangement, where L is a simple line in $\mathcal{L}$, then we get the following corollary:

Corollary 3.5. *We have that:*

$$\pi_1(\mathbb{CP}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}') \oplus \mathbb{Z}.$$

Proof. Let $\mathcal{L}' = \mathcal{L}'' \cup L_1$ where L_1 is any line in $\mathcal{L}$. Then:

$$
\begin{aligned}
\pi_1(\mathbb{CP}^2 - \mathcal{L}) \quad &= \quad \pi_1(\mathbb{CP}^2 - (\mathcal{L}'' \cup L \cup L_1)) \\
&\overset{\text{Prop. 3.4}}{\cong} \quad \pi_1(\mathbb{CP}^2 - (\mathcal{L}'' \cup L_1)) \oplus \pi_1(\mathbb{CP}^2 - (L \cup L_1)) \\
&\cong \quad \pi_1(\mathbb{CP}^2 - (\mathcal{L}'' \cup L_1)) \oplus \mathbb{Z} \\
&= \quad \pi_1(\mathbb{CP}^2 - \mathcal{L}') \oplus \mathbb{Z}
\end{aligned}
$$

$\square$

3.3. **Arrangements with at most** 6 **lines.** Fan ([Fa1], [Fa2]) has classified the arrangements with 6 lines which have more than two multiple points, and proved that in these cases the projective fundamental group is determined by the signature of the arrangement.

Together with Theorem 3.1, he proves that the projective fundamental group of a line arrangement with at most 6 lines depends only on its signature [Fa2]. This classification is repeated here at Subsection 6.3.

4. Actions on wiring diagrams

Fix a signature $S = [2^{n_2}3^{n_3}...]$. Denote by W_S the set of all lists of Lefschetz pairs with that given signature, for which the associated wiring diagram has the unique intersection property (see Subsection 2.4). Note that the number of lines and intersection points is determined by S (there are $p = \sum n_k$ points, and the number of lines ℓ is determined by Equation (1)).

In this section, we shortly describe two equivalence relations and three actions on W_S or its quotient sets which were introduced in [GTV]. The actions are motivated by the natural isometries of the plane: reflection (denoted by τ), rotation (μ) and 'shift of infinity' (σ). In [GTV] it is shown that these equivalence relations and actions preserve the incidence lattice and the affine and projective fundamental groups.

We say that two Lefschetz pairs $[a, b]$ and $[c, d]$ are *disjoint* if the integral segments $[a, b]$ and $[c, d]$ are disjoint.

Two lists of Lefschetz pairs $\mathbf{a}, \mathbf{b} \in W_S$ are *equivalent*, denoted by $\mathbf{a} \equiv \mathbf{b}$, if it is possible to reach from $\mathbf{a}$ to $\mathbf{b}$ by switching adjacent disjoint pairs. This is easily seen to be an equivalence relation on W_S.

Next, we define the relation $\overset{\triangle}{=}$. In order to simplify notation, we write $c + [a, b]$ for the Lefschetz pair $[c + a, c + b]$. Also, if $\mathbf{a}$ is a list of Lefschetz pairs, by $c + \mathbf{a}$ we mean adding c to all the pairs in $\mathbf{a}$. Suppose $1 \le c \le c + i \le c + t \le \ell$. We define the following two lists of Lefschetz pairs:

$$\begin{aligned}
\triangle_i^{(t)} &= ([i, i+1], \cdots, [t-1, t], [0, t-1], \\
&\quad [t-1, t], \cdots, [t-i, t-i+1]), \\
\nabla_i^{(t)} &= ([i-1, i], \cdots, [0, 1], [1, t], \\
&\quad [0, 1], \cdots, [t-1-i, t-i]).
\end{aligned}$$

Now, we say that $\mathbf{a} \overset{\triangle}{=} \mathbf{b}$, if there is a chain of replacements of $c + \triangle_i^{(t)}$ by $c + \nabla_i^{(t)}$, or *vice versa*, which goes from $\mathbf{a}$ to $\mathbf{b}$.

We move to define the first action, τ. If a line arrangement $\mathcal{L}$ induces a list $\mathbf{a}$ of Lefschetz pairs, then reflection with respect to a line perpendicular to the guiding line will invert the order of the pairs in $\mathbf{a}$. More generally, we define τ as follows:

Definition 4.1. *For* $\mathbf{a} = ([a_1, b_1], [a_2, b_2], \cdots, [a_p, b_p]) \in W_S$*, let*

$$\tau(\mathbf{a}) := ([a_p, b_p], [a_{p-1}, b_{p-1}], \cdots, [a_1, b_1]).$$

Note that if $\mathbf{a} \equiv \mathbf{b}$, then $\tau(\mathbf{a}) \equiv \tau(\mathbf{b})$.

The next action, μ, is motivated by the behavior of the induced wiring diagram when rotating the line arrangement. We decompose the list of pairs $\mathbf{a}$ into three disjoint sublists L_+, L_- and L_0. For this, we view every $[a_i, b_i]$ as a permutation acting on the indices $\{1, \cdots, \ell\}$, where $[a_i, b_i]$ sends every $a_i \le t \le b_i$ to $a_i + b_i - t$, and leaves the other indices fixed.

Now, set $x = 1$ and $L_+ = L_- = L_0 = \emptyset$. For each $i = 1, \ldots, p$, act as follows. If $a_i \le x \le b_i$, add $[a_i, b_i]$ to the list L_0, and set $x = a_i + b_i - x$. If $x > b_i$, add $[a_i, b_i]$ to the list L_-, and if $x < a_i$, add $[a_i, b_i]$ to the list L_+. Continue to the next value of i.

During this procedure, the line numbered 1 at ∞ always carries the local index x. As a result, whenever $a_i \le x \le b_i$, we actually have $x = a_i$ (since otherwise the line with local number a_i would have to intersect the first line at some point before, and again at point number i, contradicting the unique intersection property).

If follows that the intersection points which were combined as L_0 all lay on the first line, the points in the list L_+ are above that line, and the ones in L_- are below it.

It is now obvious that we can reorder $\mathbf{a}$ into an equivalent list $\mathbf{a}'$, which can be written as an *ordered* union $\mathbf{a}' = L_+ \cup L_0 \cup L_-$. Thus we can define an action μ as follows.

Definition 4.2. *Let* $\mathbf{a} \in W_S$ *be a list of Lefschetz pairs,*

$$\mathbf{a} = ([a_1, b_1], [a_2, b_2], \cdots, [a_n, b_n]).$$

Decompose $\mathbf{a} \equiv L_+ \cup L_0 \cup L_-$ *by the procedure described above, so that we can write:*

$$L_+ = ([a_1, b_1], \cdots, [a_u, b_u]),$$

$$L_0 = ([a_{u+1}, b_{u+1}], \cdots, [a_v, b_v]),$$

$$L_- = ([a_{v+1}, b_{v+1}], \cdots, [a_n, b_n]).$$

where $1 \le u < v \le n$.

We subtract one from the indices in L_+, invert L_0 and add one to the indices in L_-; $\mu(\mathbf{a})$ is defined by:

$$
\begin{aligned}
\mu(\mathbf{a}) \quad := \quad &([a_1 - 1, b_1 - 1], \cdots, [a_u - 1, b_u - 1], \\
&[a_v, b_v], [a_{v-1}, b_{v-1}], \cdots, [a_{u+1}, b_{u+1}], \\
&[a_{v+1} + 1, b_{v+1} + 1], \cdots, [a_n + 1, b_n + 1]).
\end{aligned}
$$

Since the decomposition $\mathbf{a} \equiv L_+ \cup L_0 \cup L_-$ is unique, it is obvious that for $\mathbf{a} \equiv \mathbf{b}$ we get $\mu(\mathbf{a}) = \mu(\mathbf{b})$.

Finally, denote by J the inversion permutation: $J(i) = \ell+1-i$, $\quad 1 \leq i \leq \ell$.

Definition 4.3. *Let $\mathbf{a} = ([a_1, b_1], [a_2, b_2], \cdots, [a_p, b_p]) \in W_S$. Then:*

$$
\sigma(\mathbf{a}) := ([a_2, b_2], [a_3, b_3], \cdots, [a_p, b_p], [J(b_1), J(a_1)]).
$$

Note that σ does not preserve equivalence classes of $\equiv$.

We say that two wiring diagrams $\mathbf{a}$ and $\mathbf{b}$ are *similar* if there is a chain of wiring diagrams from $\mathbf{a}$ to $\mathbf{b}$, such that in each step we move to an equivalent wiring diagram (with respect to $\equiv$ or $\stackrel{\Delta}{=}$), or act with τ, μ or σ. Obviously, this is again an equivalence relation.

The following theorem is proven in [GTV]:

Theorem 4.4. *Similar wiring diagrams have the same incidence lattice, and the same affine and projective fundamental groups.*

In the next section we describe an algorithm used to enumerate and find representatives of all the similarity classes for a given signature, for all the signatures of up to 8 points. The detailed results are then given in sections 6–8.

5. CLASSIFICATION OF SIMILARITY CLASSES

As explained in previous sections, our aim is to classify the wiring diagrams of up to 8 wires (which satisfy the unique intersection property) up to similarity (which is the transitive closure of $\equiv, \stackrel{\Delta}{=}$ and the relations induced by the actions τ, μ and σ). This classification is accomplished along the following lines.

(1) First, we find the possible signatures for a line arrangement with a given number of lines. For this we only need to solve Equation (1) for integral $n_2, n_3, \cdots$, and this is easy when ℓ is small. We use Lemma 5.1 below to rule out some impossible signatures.

(2) For every fixed signature, we produce a list of one representative from each similarity class. Details on this crucial step are given in Subsection 5.2.
(3) Next, we find the incidence lattice of every similarity class. It is common to discover that the few dozens of similarity classes have very few lattice structures. The mechanism used for this part is described in Subsection 5.3.
(4) In the final step, we check that every two similarity classes which have isomorphic lattices, also have the same affine and projective fundamental groups. This test was performed using the software package testisom. Details are given in Subsections 5.4 and 5.5.

The reader will observe that our procedure classifies wiring diagrams, and not only line arrangements. However, by a result of Goodman and Pollack [GP], the minimal wiring diagram with the unique intersection property which is not induced by a line arrangement has 9 wires.

5.1. **Signatures of line arrangements.**

The following lemma shows that many solutions to $\sum_{k \geq 2} n_k \binom{k}{2} = \binom{\ell}{2}$ do not occur as signatures of a line arrangement:

Lemma 5.1. *Let $[2^{n_2} 3^{n_3} \cdots]$ be a signature of a wiring diagram with ℓ pseudolines, and assume there is a point in which $\ell - c$ lines meet. Then:*

$$\sum n_k \binom{k-1}{2} \leq \binom{c}{2} + \binom{\ell - c}{2}$$

Proof. Let P be the point of multiplicity $\ell - c$, and consider the remaining c lines. If they form a diagram with signature $[2^{m_2} 3^{m_3} \cdots]$ (so that $\sum m_k \binom{k}{2} = \binom{c}{2}$), then in the original diagram the multiplicity of every point cannot be larger by more than 1. Since we now count the point P too, we have that $\sum n_k \binom{k-1}{2} \leq \binom{c}{2} + \binom{\ell-c}{2}$. $\square$

If we do not count the point in which $\ell - c$ lines meet, then for $c = 1$, the lemma gives $n_3 + \cdots \leq 0$, so the signature is $[2^{\ell-1}(\ell - 1)^1]$. For $c = 2, 3, 4$, the inequalities are $n_3 + \cdots \leq 1$, $n_3 + 3n_4 + \cdots \leq 3$ and $n_3 + 3n_4 + 6n_5 + \cdots \leq 6$, respectively.

5.2. **Computer enumeration of similarity classes.**

Fix a signature S satisfying Equation (1), for example $[2^{16} 3^4]$. (this signature has one of the largest spaces $|W_S/\equiv|$ among the signatures for which this size was computed).

The number of lines and intersection points is determined by S; in the example, there are $\ell = 8$ lines and $p = 20$ intersection points. We

want to list all the wiring diagrams with signature S, up to the action of τ, μ, σ and the equivalence relations $\equiv, \overset{\Delta}{=}$. We actually intend to generate a list, in which every element is minimal in its similarity class (with respect to the lexicographical order).

Recall that to a wiring diagram $\mathbf{a}$, we associate a list $([a_1, b_1], \cdots, [a_p, b_p])$ of Lefschetz pairs, where the multiset of differences $b_i - a_i$ gives the signature. Viewing each pair $[a, b]$ as the permutation sending $a \leq x \leq b$ to $a + b - x$, the pseudoline numerated $1 \leq i \leq \ell$ at $+\infty$, becomes $[a_p, b_p] \cdots [a_1, b_1](i)$ at $-\infty$. By Lemma 2.3, $\mathbf{a}$ has the unique intersection property if and only if $[a_p, b_p] \cdots [a_1, b_1] = J$, where $J = [1, \ell]$ sends $x \mapsto \ell + 1 - x$.

A naive approach to finding all the possible wiring diagrams would be to try every possible lists of Lefschetz pairs $([a_1, b_1], \cdots, [a_p, b_p])$, and keep only those lists for which the product is J. This is not practical, as there are too many products to check (in our example $S = [2^{16}3^4]$, there are $\binom{20}{4} \cdot 7^{16} \cdot 6^4 = 10^{20.3}$ such products). Instead, we use a method known as 'meet in the middle' (see e.g. [MH]). This method is used to solve the following type of problems: $A_1, \ldots, A_p$ are given (small, say $|A_i| = 5$ or $|A_i| = 7$) sets of permutations, and we want to list all the solutions to $\pi_p \pi_{p-1} \ldots \pi_1 = J$, $\pi_i \in A_i$, for some fixed permutation J. The method trades time for memory in the following way. Instead of going through all the $|A_1| \cdots |A_p|$ possibilities for p-tuples $(\pi_1, \cdots, \pi_p)$, we fix some $1 < p_0 < p$, and generate a sorted list of all the products $\pi_{p_0} \cdots \pi_1$ ($\pi_i \in A_i$). After this half is finished, we construct all the products $\pi_{p_0+1}^{-1} \cdots \pi_p^{-1} J$ and look for each of them in the list generated before. If $\pi_{p_0} \cdots \pi_1 = \pi_{p_0+1}^{-1} \cdots \pi_p^{-1} J$, then $(\pi_1, \cdots, \pi_p)$ is a solution to the original problem.

If, for example, the sets A_i all have the same size c and we take $p_0 = p/2$, then this procedure takes $2c^{p/2}$ operations and memory of $c^{p/2}$ permutations, instead of c^p operations (and no memory) in the naive way. In general, if memory of $c^{p/2}$ is available, it would be best to take $p_0 \sim p/2$.

In our problem, we would like to use the sets $A_i = \{[1, c], [2, c + 1], \cdots, [\ell+1-c, \ell]\}$, where c is the number of lines intersecting in the ith point. However, we do not know in advance how many lines intersect in each point: the signature only determines how many points of each type there are. To overcome this obstacle, we go through all the possible ways to break the given signature S into a sum of signatures $S = S_0 + S_1$, where S_0 carries p_0 intersection points, and S_1 the remaining $p - p_0$ points. For each dissection $S = S_0 + S_1$, we go through the different choices of the numbers of lines meeting in every point; then

the sets A_i are fixed. We continue as in the standard method, and collect the solutions from all the possible dissections.

All this suggests the following algorithm: first use the 'meet in the middle' method (with certain variations) to list all the wiring diagrams with signature S, then somehow pick the minimal element from each similarity class. However, this method is still not practical due to the huge number of possible wiring diagrams. For $S = [2^{16}3^4]$ there are about $10^{10.6}$ such classes (this was not directly computed; the way we got this estimate is explained below).

Luckily, we are interested in the list of similarity classes and not in the full list of wiring diagrams.

Fixing $1 \le p_0 \le p$, we define the following equivalence relation, denoted $\equiv_{\frac{1}{2}}$: given a wiring diagram with p intersection points, we can uniquely write it as a composition of two wiring diagrams on the same wires, $\mathbf{a} = \mathbf{a}_2\mathbf{a}_1$, where $\mathbf{a}_1$ contains the first p_0 intersection points, and $\mathbf{a}_2$ the other $p - p_0$. Decompose $\mathbf{a} = \mathbf{a}_2\mathbf{a}_1$ and $\mathbf{b} = \mathbf{b}_2\mathbf{b}_1$. Then $\mathbf{a} \equiv_{\frac{1}{2}} \mathbf{b}$ if and only if $\mathbf{a}_1 \equiv \mathbf{b}_1$ and $\mathbf{a}_2 \equiv \mathbf{b}_2$. Obviously, $\equiv_{\frac{1}{2}}$ is an equivalence relation (with smaller classes than those of $\equiv$, in general).

We use $<$ to denote the lexicographical order between wiring diagrams (*i.e.* the associated lists of Lefschetz pairs). Suppose $\mathbf{a} = \mathbf{a}_2\mathbf{a}_1 \in W_S$ is a wiring diagram; if $\mathbf{a}_2'\mathbf{a}_1$ is also a wiring diagram with $\mathbf{a}_2' \equiv \mathbf{a}_2$ and $\mathbf{a}_2 < \mathbf{a}_2'$, then $\mathbf{a}_2\mathbf{a}_1 < \mathbf{a}_2'\mathbf{a}_1$. We may thus keep only the minimal among all $\equiv$-equivalent halves $([a_1, b_1], \cdots, [a_{p_0}, b_{p_0}])$ which multiply to the same permutation. The same argument holds of course for the second half of the wiring diagrams.

To summarize, this improved 'meet in the middle' mechanism produces a list of elements, which are minimal representatives in their classes with respect to $\equiv_{\frac{1}{2}}$. Every wiring diagram in W_S is $\equiv_{\frac{1}{2}}$-equivalent to some element in this list.

Minimality in both halves does not guarantee that an element is minimal in its $\equiv$-class. Indeed, such $\mathbf{a}$ is not minimal exactly when $[a_{p_0+1}, b_{p_0+1}] < [a_{p_0}, b_{p_0}]$. We simply erase these elements, and the result is a list ω_S of minimal representatives for all the $\equiv$-classes. For the signature $S = [2^{16}3^4]$, there are 3317776 $\equiv$-classes (and 8289348 $\equiv_{\frac{1}{2}}$-classes).

Given this list, it is possible to estimate the size of W_S as follows. We randomly choose some of the representatives, compute the size of their $\equiv$-class (the complexity is linear in the size of the class), and use the average to estimate the average size of all the classes. This average is then multiplied by the number of classes.

Let $[\mathbf{a}]$ denotes the $\equiv$-class of $\mathbf{a}$. Recall that $\mathbf{a}, \mathbf{b} \in \omega_S$ are similar if it is possible to reach from an element of $[\mathbf{a}]$ to an element of $[\mathbf{b}]$ by action of τ, μ or σ, or by replacing a wiring diagram $\mathbf{c}$ by $\mathbf{c}' \equiv \mathbf{c}$ or $\mathbf{c}' \overset{\triangle}{=} \mathbf{c}$.

For $\mathbf{a} \in W_S$, we denote by $\mathbf{a}_0$ the minimal element in $[\mathbf{a}]$ (with respect to the lexicographical order). For every $\mathbf{a} \in \omega_S$, we compute $\tau(\mathbf{a})_0$ and $\mu(\mathbf{a})_0$ which are in ω_S. We also compute the set $\sigma([\mathbf{a}])_0$: σ is not well-defined on $W_S/\equiv$, but for $\mathbf{a}' \equiv \mathbf{a}$, $[\sigma(\mathbf{a}')]$ depends only on the leftmost pair in $\mathbf{a}'$. Thus, $\sigma([\mathbf{a}])_0$ is of size no more than p (but usually much less). Finally, we find $\mathbf{b}_0$ for every $\mathbf{b} \overset{\triangle}{=} \mathbf{a}$. This is done by locating lists of pairs of the form $c + \triangle_i^{(t)}$ or $c + \nabla_i^{(t)}$, which are consecutive in some member of $[\mathbf{a}]$, and replacing $c + \triangle_i^{(t)}$ by $c + \nabla_i^{(t)}$, or *vice versa*.

We generate a graph on ω_S as the set of vertices, where $\mathbf{a} \in \omega_S$ is connected to all the elements $\tau(\mathbf{a})_0, \mu(\mathbf{a})_0, \sigma([\mathbf{a}])_0$ and $\mathbf{b}_0$ for $\mathbf{b} \overset{\triangle}{=} \mathbf{a}'$, $\mathbf{a}' \in [\mathbf{a}]$.

It remains only to list the minimal representatives for the connected components of this graph by standard depth-first search. The result is the required list of minimal representatives of the similarity classes.

The running time of this algorithm depends of course on the size of W_S. The worst case for arrangements of up to 8 lines was $[2^{19}3^3]$, which took several CPU hours on a medium-size Unix machine; the size of W_S is about $10^{11.75}$, and there were 5 similarity classes.

To demonstrate the relative influence of the relation $\overset{\triangle}{=}$ and the actions μ, τ, σ, we give below the number of classes of $W_S/\equiv$ (for $S = [2^{16}3^4]$) under the various sets of relations. For example, the fourth line tells us that under the transitive closure of $\mu, \overset{\triangle}{=}$ and $\equiv$, there are 10948 classes.

σ	τ	μ	$\overset{\Delta}{=}$	Number of classes
-	-	-	-	3317776
-	-	-	+	306328
-	-	+	-	207361
-	-	+	+	10948
-	+	-	-	1658965
-	+	-	+	125186
-	+	+	-	103719
-	+	+	+	5543
+	-	-	-	35100
+	-	-	+	460
+	-	+	-	723
+	-	+	+	18
+	+	-	-	17644
+	+	-	+	259
+	+	+	-	723
+	+	+	+	18

The redundancy of τ in the presence of σ and μ is explained in [GTV, Cor. 5.3].

5.3. **Lattice comparison.** Let $\mathbf{a} = ([a_1, b_1], \cdots, [a_p, b_p])$ be a list of Lefschetz pairs. The incidence lattice of $\mathbf{a}$ is determined by the sets of intersection points on each of the wires. To compute the lattice, we compute the global numeration of the wires at each point, by applying the permutation $[a_i, b_i] \cdots [a_1, b_1]$ on the list $1, \cdots, \ell$ for every point i. In what follows, we view the incidence lattice as the bipartite graph whose vertices are the lines and points.

The problem of checking if two bipartite graphs are isomorphic is known to be difficult in general (see [GJ, p. 285]). However (as is often the case), one can get quite good results by heuristic methods.

We hold a graph as a matrix M in the usual way: $M_{ij} = 1$ if the point number j is on line number i, and $M_{ij} = 0$ otherwise.

The idea is to sort the rows and the columns of the matrix alternatively. After doing it several times, we compare the two sorted matrices. If we get equal matrices, the corresponding lattices are equivalent.

Using various sorting orders, we are able to classify the lattices into few families. Then, we manually check if these families are indeed distinct.

5.4. **Computation of the fundamental group.** The computation of the fundamental group of the complement of real line arrangements is done in two steps: the Moishezon-Teicher algorithm computes the

skeletons associated to the arrangement, and the van Kampen theorem produces from the skeletons a finite presentation of the affine fundamental group. One more relation gives the projective fundamental group.

The theoretical background for both steps can be found in [MoTe], [vK] and [GaTe]. Here we only sketch the ideas from an algorithmic point of view. Full details are given in Section 3 of [GTV].

Let $\mathbf{a} = ([a_1, b_1], \cdots, [a_p, b_p])$ be a list of Lefschetz pairs. In order to compute the skeleton s_i associated to the ith intersection point $[a_i, b_i]$, we start with the configuration in Figure 3, in which segments connect the points from a_i to b_i.
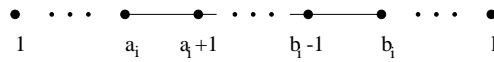
FIGURE 3

and apply the Lefschetz pairs $[a_{i-1}, b_{i-1}], \cdots, [a_1, b_1]$. A Lefschetz pair $[a_j, b_j]$ acts by rotating the region from a_j to b_j by 180° counterclockwise without affecting any other points.

For example, consider the list $([2, 3], [2, 4], [4, 5], [1, 3], [3, 4])$. The initial skeleton for $[3, 4]$ is given in Figure 4. Applying $[1, 3]$ and then
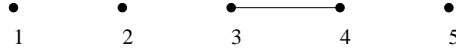
FIGURE 4

$[4, 5]$, we get the skeleton in Figure 5. Then, applying $[2, 4]$ yields the
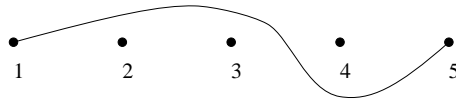
FIGURE 5

skeleton of Figure 6, and finally, acting with $[2, 3]$ we get the skeleton in Figure 7.
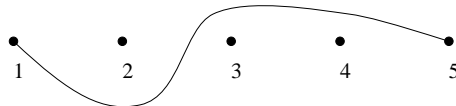
FIGURE 6
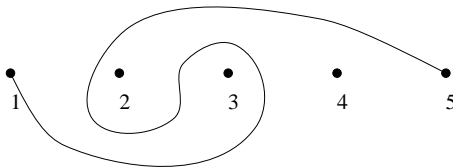
FIGURE 7

Since we had to compute hundreds of fundamental groups, this procedure had to be programmed. The algorithm was implemented as follows: we decode a path as a list of the points near which the path passes, with special marks for 'pass above' and 'pass below'. For example, the path in Figure 7 is decoded as $1°2^-3^-3^+2^-2^+3^+4^+5°$. Details on the implementation can be found in [GKT] and [Ga].

The fundamental group is generated by $\Gamma_1, \ldots, \Gamma_\ell$, where Γ_j corresponds to loop around the ith point which is based on a fixed point outside the arrangement.

Assume the skeleton s_i is induced by a simple intersection point. Then the induced relation on the Γ_j is computed as follows. We cut the skeleton just after its connection to its start point, and denote the associated generator by a_1. The second element is computed by going from the starting point around the end point: whenever the path passes above a point we multiply by the generator Γ_j associated to that point or its inverse, depending on the direction of the path (we ignore the points which we pass from below). When we reach the end point, we add the generator associated to this point, and then go all the way back by adding the corresponding inverse elements. This results with an element a_2, and the relation induced by s_i is $a_1a_2 = a_2a_1$.

For example, the skeleton in Figure 5 induces the relation

$$\Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1} \cdot \Gamma_5 = \Gamma_5 \cdot \Gamma_3\Gamma_2\Gamma_1\Gamma_2^{-1}\Gamma_3^{-1}.$$

Details can be found in [GaTe].

In case the skeleton s_i corresponds to a multiple intersection point, we get a set of relations of the form $a_c \ldots a_2 a_1 = a_1 a_c \ldots a_2 = \cdots = a_{c-1} \ldots a_1 a_c$, where a_r is computed from the (same) starting point around the r-th point, as described above.

To obtain the projective fundamental group, we add the relation $\Gamma_\ell \cdots \Gamma_2 \Gamma_1 = 1$.

5.5. **Checking isomorphisms between fundamental groups.** The problem of testing if two groups (presented in terms of generators and relations) are isomorphic is undecidable (see [Ro]). However, the fact that the problem is undecidable in general does not prevent one from

trying to solve it in concrete cases. Indeed, there is a software package called testisom, written by D. Holt and S. Rees, which attempts to prove or disprove that two given groups are isomorphic. It tries to guess an isomorphism using the Knuth-Bendix algorithm (which generates part of Cayley graph of the groups), or prove the groups are non-isomorphic by computation of Abelian sections, mainly of the lower central series, or counting projections onto certain small groups.

For full description of this package, see [EHR],[HR].

6. Arrangements with up to 6 lines

In this section we classify the arrangements with up to 6 lines, up to lattice isomorphism.

6.1. **Possible signatures for $\ell \leq 6$ lines.** The following proposition lists the different possible signatures for arrangements with up to 6 lines.
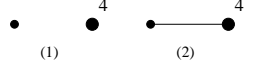
Proposition 6.1. *The possible signatures for arrangements with up to 6 lines (sorted by the number of intersection points) are:*

(1) $[2^1]$ *(for 2 lines),*
(2) $[3^1]$, $[2^3]$ *(for 3 lines),*
(3) $[4^1]$, $[2^3 3^1]$, $[2^6]$ *(for 4 lines),*
(4) $[5^1]$, $[2^4 4^1]$, $[2^4 3^2]$, $[2^7 3^1]$, $[2^{10}]$ *(for 5 lines), and*
(5) $[6^1]$, $[2^5 5^1]$, $[2^3 3^4]$, $[2^6 3^1 4^1]$, $[2^6 3^3]$, $[2^9 4^1]$, $[2^9 3^2]$, $[2^{12} 3^1]$ *and* $[2^{15}]$ *(for 6 lines).*

Proof. There are some solutions to $\sum_{k \geq 2} n_k \binom{k}{2} = \binom{\ell}{2}$ except for the ones listed above. For $\ell = 4$ we have $[3^2]$, which is forbidden by the case $c = 1$ of Lemma 5.1. For $\ell = 5$, we get $[2^1 3^1 4^1]$ and $[2^1 3^3]$. The first is forbidden by Lemma 5.1, case $c = 1$, and the second by Lemma 5.1, case $c = 2$.

For $\ell = 6$, we get $[2^2 3^1 5^1]$, $[3^1 4^2]$, $[2^3 4^2]$, $[3^3 4^1]$, $[2^3 3^2 4^1]$ and $[3^5]$. The first five options are forbidden by Lemma 5.1. As for $[3^5]$, this is impossible since then a line L would meet five other lines, two in each intersection point; but 2 does not divide 5. □

In order to present the results of the classification, we use diagrams based on the graphs $G(\mathcal{L})$ of multiple points, defined in Section 3. In these diagrams we omit all the simple points and the lines on which there is at most one multiple points (these are easily put back in place if one wants to reconstruct the arrangements from our diagrams). We denote a multiple point of multiplicity 3 by a black dot, and a multiple point of higher multiplicity by a bigger dot with a number indicating

the multiplicity. Intersection points which are not marked by a dot are simple. If more than two points belong to the same line, we draw the line thicker than the usual. From this graph, one can recover the lattice of the arrangement, and construct explicitly the corresponding line arrangement. For example, the arrangements of Figure 2 are represented by the graphs given in Figure 8.

(1)   (2)

FIGURE 8

If there are at most two multiple points, then the affine and projective fundamental groups can be computed for every lattice by Theorems 3.2 and 3.1. These groups are given in Subsection 6.2. For the other signatures we do not give the groups, even though their presentations were computed for all the similarity classes. The classification of this case is given in Subsection 6.3. We keep the same structure in Sections 7 and 8.

6.2. **Signatures with at most two multiple points.** The following table summarizes the possible lattices for signatures in which there are at most two multiple points. We give only the projective fundamental group, and note that by Theorems 3.1 and 3.2, in these cases the affine fundamental group is given by

$$(2) \qquad \pi_1(\mathbb{C}^2 - \mathcal{L}) \cong \pi_1(\mathbb{CP}^2 - \mathcal{L}) \oplus \mathbb{Z}.$$

ℓ	Signature	Possible lattices	Projective fundamental group
2	$[2^1]$	no multiple points	$\mathbb{Z}$
3	$[3^1]$	•	$\mathbb{F}_2$
3	$[2^3]$	no multiple points	$\mathbb{Z}^2$
4	$[4^1]$	•⁴	$\mathbb{F}_3$
4	$[2^3 3^1]$	•	$\mathbb{Z} \oplus \mathbb{F}_2$
4	$[2^6]$	no multiple points	$\mathbb{Z}^3$
5	$[5^1]$	•⁵	$\mathbb{F}_4$
5	$[2^4 4^1]$	•⁴	$\mathbb{Z} \oplus \mathbb{F}_3$
5	$[2^4 3^2]$	•——•	$\mathbb{F}_2 \oplus \mathbb{F}_2$
5	$[2^7 3^1]$	•	$\mathbb{Z}^2 \oplus \mathbb{F}_2$
5	$[2^{10}]$	no multiple points	$\mathbb{Z}^4$
6	$[6^1]$	•⁶	$\mathbb{F}_5$
6	$[2^5 5^1]$	•⁵	$\mathbb{Z} \oplus \mathbb{F}_4$
6	$[2^6 3^1 4^1]$	•⁴——•	$\mathbb{F}_2 \oplus \mathbb{F}_3$
6	$[2^9 4^1]$	•⁴	$\mathbb{Z}^2 \oplus \mathbb{F}_3$
6	$[2^9 3^2]$	• • •——• (1) (2)	$\mathbb{Z} \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$
6	$[2^{12} 3^1]$	•	$\mathbb{Z}^3 \oplus \mathbb{F}_2$
6	$[2^{15}]$	no multiple points	$\mathbb{Z}^5$

6.3. **Signatures with more than two multiple points.** In the following table, we summarize the possible lattices for signatures with more than two multiple points:

Signature	Possible lattices	$|W_S/\equiv|$	Similarity classes
$[2^3 3^4]$	(square with diagonals)	16	1
$[2^6 3^3]$	(triangle)	304	2

7. Arrangements with 7 lines

In this section we classify the arrangements with 7 lines up to lattice isomorphism.

7.1. Possible signature for $\ell = 7$.

Proposition 7.1. *The possible signatures for arrangements with* 7 *lines are:* $[7^1]$, $[2^6 6^1]$, $[2^3 3^6]$, $[2^8 3^1 5^1]$, $[2^6 3^3 4^1]$, $[2^9 4^2]$, $[2^6 3^5]$, $[2^9 3^2 4^1]$, $[2^{11} 5^1]$, $[2^9 3^4]$, $[2^{12} 3^1 4^1]$, $[2^{12} 3^3]$, $[2^{15} 4^1]$, $[2^{15} 3^2]$, $[2^{18} 3^1]$, $[2^{21}]$.

Proof. Except for the ones listed above, the only solutions to $\sum_{k \geq 2} n_k \binom{k}{2} = \binom{7}{2}$ are: $[4^1 6^1]$, $[3^2 6^1]$, $[2^3 3^1 6^1]$ which are forbidden by the case $c = 1$ of Lemma 5.1; $[2^1 5^2]$, $[2^2 3^1 4^1 5^1]$, $[2^5 4^1 5^1]$, $[2^2 3^3 5^1]$, $[2^5 3^2 5^1]$ which are forbidden by the case $c = 2$ of this lemma; $[3^1 4^3]$, $[2^3 4^3]$, $[3^3 4^2]$, $[2^3 3^2 4^2]$, $[2^6 3^1 4^2]$, $[3^5 4^1]$, $[2^3 3^4 4^1]$ and $[3^7]$.
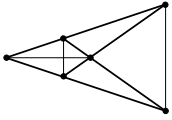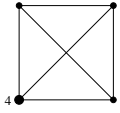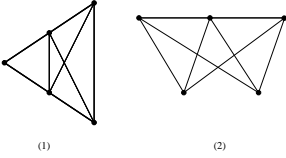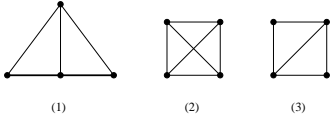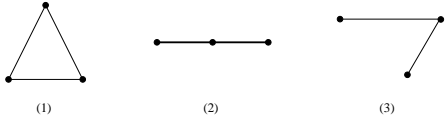
The first seven options are excluded by the case $c = 3$ of the lemma, since we must have $n_3 + 3n_4 \leq 3$ (note that here n_4 counts the points of multiplicity 4 beyond the first one). As for $[3^7]$, this is the projective geometry on 7 points, which is well known not to be embedded in the affine plane. $\square$

We now describe the different line arrangements in details. We use the encoding of lattices into diagrams, described in section 6.

7.2. Signatures with at most two multiple points.
The following table lists the possible lattices for the signatures of arrangements with 7 lines, if there are at most two multiple points. We give the corresponding projective fundamental groups, which are computed using Theorem 3.1. From this one can compute the affine groups by Equation (2).

Signature	Possible lattices	Projective fundamental group
$[7^1]$		$\mathbb{F}_6$
$[2^6 6^1]$		$\mathbb{Z} \oplus \mathbb{F}_5$
$[2^8 3^1 5^1]$		$\mathbb{F}_2 \oplus \mathbb{F}_4$
$[2^9 4^2]$		$\mathbb{F}_3 \oplus \mathbb{F}_3$
$[2^{11} 5^1]$		$\mathbb{Z}^2 \oplus \mathbb{F}_4$
$[2^{12} 3^1 4^1]$		$\mathbb{Z} \oplus \mathbb{F}_2 \oplus \mathbb{F}_3$
$[2^{15} 4^1]$		$\mathbb{Z}^3 \oplus \mathbb{F}_3$
$[2^{15} 3^2]$		$\mathbb{Z}^2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$
$[2^{18} 3^1]$		$\mathbb{Z}^4 \oplus \mathbb{F}_2$
$[2^{21}]$	no multiple points	$\mathbb{Z}^6$

7.3. **Signatures with more than two multiple points.** In the following table, we summarize the possible lattices for signatures with more than two multiple points.

Signature	Possible lattices	$|W_S/\equiv|$	Similarity classes
$[2^3 3^6]$		28	1
$[2^6 3^3 4^1]$		448	1
$[2^6 3^5]$		770	2
$[2^9 3^2 4^1]$		3864	3
$[2^9 3^4]$		9534	6
$[2^{12} 3^3]$		45458	4

8. Arrangements with 8 lines

Here we classify the arrangements with 8 lines up to lattice isomorphism.

8.1. **Possible signatures for $\ell = 8$.**

Proposition 8.1. *The following are all the possible signatures for arrangements with 8 lines:* $[8^1]$, $[2^7 7^1]$, $[2^4 3^6 4^1]$, $[2^{10} 3^1 6^1]$, $[2^7 3^3 4^2]$, $[2^9 3^3 5^1]$, $[2^7 3^5 4^1]$, $[2^{12} 4^1 5^1]$, $[2^{13} 6^1]$, $[2^7 3^7]$, $[2^{10} 3^2 4^2]$, $[2^{12} 3^2 5^1]$, $[2^{10} 3^4 4^1]$, $[2^{10} 3^6]$, $[2^{13} 3^1 4^2]$, $[2^{15} 3^1 5^1]$, $[2^{13} 3^3 4^1]$, $[2^{13} 3^5]$, $[2^{16} 4^2]$, $[2^{18} 5^1]$, $[2^{16} 3^2 4^1]$, $[2^{16} 3^4]$, $[2^{19} 3^1 4^1]$, $[2^{19} 3^3]$, $[2^{22} 4^1]$, $[2^{22} 3^2]$, $[2^{25} 3^1]$, $[2^{28}]$.

Proof. There are 38 solutions to the equation $\sum_{k \geq 2} n_k \binom{k}{2} = \binom{8}{2}$ *except* for the above listed ones. Lemma 5.1 excludes three cases with $c = 1$,

nine cases with $c = 2$ and fifteen cases with $c = 3$. Out of the nine possibilities with a point with multiplicity 4, the lemma (with $c = 4$) excludes eight, the remaining one is $[2^{10}4^3]$. Suppose there are at least two points in which four lines meet. If they do not share a common line, the signature must be $[2^{16}4^2]$. Otherwise, there is one remaining line, which can meet at most 3 otherwise-simple intersection points, so we cannot have a third point of multiplicity 4.

The last two cases are $[2^1 3^9]$ and $[2^4 3^8]$. Assume signature of the form $[2^{n_2} 3^{n_3}]$. Every line meets seven other lines, so the number of multiple points on a line is no more than 3. Together, there are no more than $\frac{8 \cdot 3}{3} = 8$ multiple points (since each point is counted three times). This argument rules out the signature $[2^1 3^9]$. It also shows that in a diagram with signature $[2^4 3^8]$, on each line there are exactly one simple point and three multiple points. We could not find a geometric argument to rule this case out (see Remark 8.2 below), but our algorithm (Section 5.2) found no wiring diagrams for this signature.   □

Remark 8.2. *We would like to explain why it is difficult to show that there are no wiring diagrams with the signature $[2^4 3^8]$. All the arguments in Propositions 6.1, 7.1 and 8.1 are combinatorial in nature, actually showing that there is no arrangement of curves with the unique intersection property which induces the forbidden signatures (with the exception of $[3^7]$ which indeed is induced by the projective geometry on 7 points).*

However, we now show that $[2^4 3^8]$ is in fact induced by an arrangement of curves. For that, it is enough to describe a bipartite graph (on the sets C of 'curves' and P of 'points'), in which every $c_1, c_2 \in C$ 'meet' once (i.e. are connected to a single $p \in P$), and such that there are 4 points of degree (=total number of edges) 2, and 8 points of degree 3.

We let numbers denote the curves, and small Latin letters denote the points. Consider the Cayley graph of the dihedral group D_8 with respect to the presentation,

$$\langle \alpha, \beta, \gamma \mid \alpha^2 = \beta^2 = \gamma^2 = 1, \gamma\alpha\gamma = \beta\gamma\beta, \beta\alpha\beta = \alpha\gamma\alpha \rangle$$

shown in Figure 9 (with β the generator drawn horizontally).

This is a bipartite graph of degree 3, on $8 + 8$ points (in Coxeter's notation [Cox, Figs. 11,12], this is $\{8\} + \{8/3\}$). Note that the only couples of curves which do not share a common neighbor are $\{1, 8\}, \{2, 6\}, \{3, 5\}$ and $\{4, 7\}$. When we add the points connecting these couples we get the desired graph. This is the only graph with these properties.
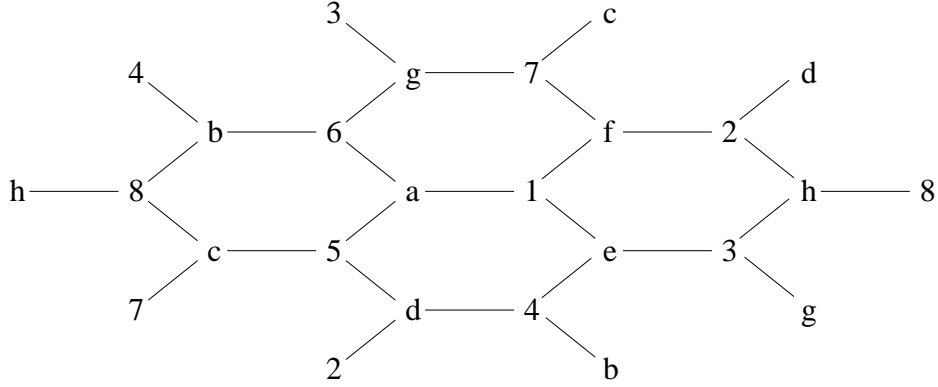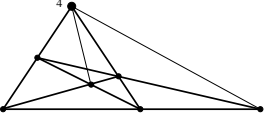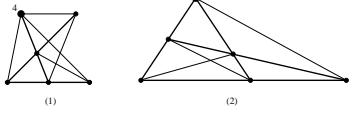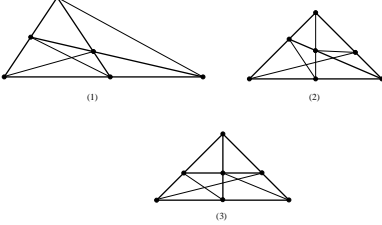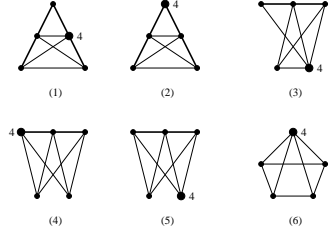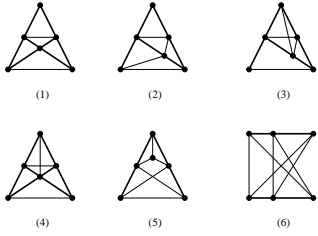
FIGURE 9

8.2. **Signatures with at most two multiple points.** We now list the possible lattices and groups for signatures with at most two multiple points for $\ell = 8$ lines. As before, we give only the projective groups, from which one can compute the affine groups by Equation (2).

Signature	Possible lattices	Projective fundamental group
$[8^1]$	$\overset{8}{\bullet}$	$\mathbb{F}_7$
$[2^7 7^1]$	$\overset{7}{\bullet}$	$\mathbb{Z} \oplus \mathbb{F}_6$
$[2^{10} 3^1 6^1]$	$\bullet\!\!-\!\!-\!\!\overset{6}{\bullet}$	$\mathbb{F}_2 \oplus \mathbb{F}_5$
$[2^{12} 4^1 5^1]$	$\overset{4}{\bullet}\!\!-\!\!-\!\!\overset{5}{\bullet}$	$\mathbb{F}_3 \oplus \mathbb{F}_4$
$[2^{13} 6^1]$	$\overset{6}{\bullet}$	$\mathbb{Z}^2 \oplus \mathbb{F}_5$
$[2^{15} 3^1 5^1]$	$\bullet\!\!-\!\!\overset{5}{\bullet}$ (1) $\quad\bullet\quad\overset{5}{\bullet}$ (2)	$\mathbb{Z} \oplus \mathbb{F}_2 \oplus \mathbb{F}_4$
$[2^{16} 4^2]$	$\overset{4}{\bullet}\!\!-\!\!\overset{4}{\bullet}$ (1) $\quad\overset{4}{\bullet}\quad\overset{4}{\bullet}$ (2)	$\mathbb{Z} \oplus \mathbb{F}_3 \oplus \mathbb{F}_3$
$[2^{18} 5^1]$	$\overset{5}{\bullet}$	$\mathbb{Z}^3 \oplus \mathbb{F}_4$
$[2^{19} 3^1 4^1]$	$\bullet\!\!-\!\!\overset{4}{\bullet}$ (1) $\quad\bullet\quad\overset{4}{\bullet}$ (2)	$\mathbb{Z}^2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_3$
$[2^{22} 4^1]$	$\overset{4}{\bullet}$	$\mathbb{Z}^4 \oplus \mathbb{F}_3$
$[2^{22} 3^2]$	$\bullet\!\!-\!\!-\!\!\bullet$ (1) $\quad\bullet\quad\bullet$ (2)	$\mathbb{Z}^3 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$
$[2^{25} 3^1]$	$\cdot$	$\mathbb{Z}^5 \oplus \mathbb{F}_2$
$[2^{28}]$	no multiple points	$\mathbb{Z}^7$

8.3. **Signatures with more than two multiple points.** In the following table, we summarize the possible lattices for signatures of $\ell = 8$ lines with more than two multiple points.

Signature	Possible lattices	≡ classes	Similarity classes
$[2^4 3^6 4^1]$		144	1
$[2^7 3^3 4^2]$		384	1
$[2^9 3^3 5^1]$		1792	2
$[2^7 3^5 4^1]$		5024	4
$[2^7 3^7]$		3920	4
$[2^{10} 3^2 4^2]$		6528	4
$[2^{12} 3^2 5]$		12096	4
$[2^{10} 3^4 4^1]$		63344	18
$[2^{10} 3^6]$		88608	19

Signature	Possible lattices	$\equiv$ classes	Similarity classes
$[2^{13}3^14^2]$		21984	4
$[2^{13}3^34^1]$		354880	22
$[2^{13}3^5]$		776064	31
$[2^{16}3^24^1]$		884864	7
$[2^{16}3^4]$		3317776	18
$[2^{19}3^3]$		7451568	5

For each signature (here, and in the previous tables), the algorithm described in Subsection 5.2 produces a list of representatives for the similarity classes. After comparing their incidence lattices (Subsection 5.3), we compute their fundamental groups and compare each pair with the same lattice. In all the cases we found that the affine and projective fundamental groups are isomorphic, hence we have the following:

Theorem 8.3. *For real arrangements of up to 8 lines, the incidence lattice determines the affine and projective fundamental groups.*

In two cases we did not have to compare all the groups directly. For the signature $S = [2^{16}3^24^1]$, in cases (2)–(4) the graph of multiple points has no cycles, so the projective group depends only on the lattice by Theorem 3.1. For the affine group, Theorem 3.2 settles case (4), and in cases (2)–(3) we did compare the appropriate groups. For case (1) there is a simple line, and then Theorem 3.5 treats the projective groups, and the special case of Proposition 3.3 the affine groups.

The other signature for which the groups were not compared is $S = [2^{19}3^3]$ (here we found the largest space $W_S/\equiv$, of size 7451568). For lattices (1)–(3), Theorem 3.5 reduces to seven lines for the projective groups and the special case of Proposition 3.3 for the affine groups. For case (4) we use Theorems 3.1 and 3.2 (alternatively, one can compute the groups using Propositions 3.3 and 3.4 since the graph of multiple points is disconnected). As a matter of fact, in lattices (2)–(4) there is a unique similarity class, so there was nothing to check.

The bottle neck of the algorithm described in Subsection 5.2 is not so much the running time, but the memory required to collect the representatives of $W_S/\equiv$. The extreme cases of $\ell = 8$ lines needed enough memory to upset our system manager, so we did not continue to $\ell = 9$.

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