GENERATORS OF CENTRAL SIMPLE p-ALGEBRAS OF DEGREE 3

UZI VISHNE

Abstract. We discuss standard pairs of generators of cyclic division p-algebras of degree p, and prove for $p = 3$ that any two Artin-Schreier elements are connected by a chain of standard pairs. This result has immediate applications to the presentations of such algebras.

Accepted to *Israel Journal of Math.*, 6/2001.

1. Introduction

Let Q be a quaternion algebra over a field F. It is well known (for example, see [2, Lemma 6.3]), that if $Q = (a, b) = (a', b')$ are two presentations of Q, then there is some $c \in F$ such that

$$(a, b) = (a, c) = (a', c) = (a', b').$$

Recently, a similar result for cyclic division algebras of degree 3 was proved by M. Rost [6]. If $A = (a, b)_3 = (a', b')_3$ are two presentations of A (where the base field contains 3-roots of unity), then there exist elements c, d, e in the base field, such that

$$(a, b)_3 \cong (a, c)_3 \cong (d, c)_3 \cong (d, e)_3 \cong (a', e)_3 \cong (a', b')_3.$$

Chains of this form were also studied, in a more general context, in [4].

If the degree of a central simple algebra is a power of the characteristic p of the base field, it is called a *p-algebra*. Standard generators of cyclic p-algebras of degree p were studied in the author's dissertation [8, Chap. 1, Sec. 4]. Theorem 4.16 there is, in a sense, a chain lemma for arbitrary p, but it requires tensoring by matrices.

In Section 2 we describe the basic properties of standard pairs of generators and related definitions are given. We discuss the notion of distance between Artin-Schreier elements, and state the main result, Theorem 2.6, and the applications to presentations of cyclic p-algebras.

Date: Nov 3, 2000.

The research was done while the author held a post-doctoral position at UCL, Louvain-la-Neuve, Belgium.

We study short chains of pairs for $p=3$ in Section 3, and this is applied in Section 4 to prove Theorem 2.6. Some large subgraphs of the graph of standard pairs of generators are given in Section 5.

I am indebted to Prof. J.-P. Tignol for pointing out references [4] and [6] and for his kind hospitality during my stay at UCL.

2. Standard generators of cyclic p-algebras of degree p

Let F be a field of characteristic p, and A a central simple cyclic algebra of degree p over F (that is, $\dim_F A = p^2$). By Wedderburn's structure theorem, A is either a division algebra, or the algebra of $p \times p$ matrices over F. The basic structure theory of p-algebras is given in [1], *cf.* also [5].

It is known that A can be given the following presentation, where $a, b \in F$, $b \neq 0$:

$$A = F[x,y| \quad x^p - x = a, \quad y^p = b, \quad yxy^{-1} = x+1].$$

We call such x, y a *standard pair of generators.* Let

$$\begin{aligned} \mathrm{X}_A &= \{x \in A : \quad x^p - x \in F, \quad [F[x]\!:\!F] = p\}, \\ \mathrm{Y}_A &= \{y \in A : \quad y^p \in F^*, \qquad [F[y]\!:\!F] = p\} \end{aligned}$$

be the possible components of a standard pair of generators. The elements of X_A are called *Artin-Schreier elements* of A; every cyclic subfield of A contains such an element.

Remark 2.1. *If $x, y \in A$ satisfy $yxy^{-1} = x+1$, then x, y form a standard pair of generators, that is, $A = F[x,y]$, $x^p - x = a$ and $y^p = b$ for some $a, b \in F$.*

Proof. We first show that x, y generate A. Indeed, $F[x]$ is a separable extension of dimension p over F (with an automorphism $x \mapsto x+1$ induced by y). Note that $[y^i, x] = y^i x - x y^i = i y^i$. Now suppose $f_0 + f_1 y + \cdots + f_{p-1} y^{p-1} = 0$ for $f_i \in F[x]$. Applying the derivation by x, we get $0 = f_1 y + 2 f_2 y^2 + \cdots + (p-1) f_{p-1} y^{p-1}$. Repeating this, we get $0 = f_1 y + 2^j f_2 y^2 + \cdots + (p-1)^j f_{p-1} y^{p-1}$ for every $j = 1, \dots, p-1$. Since the Vandermonde matrix of $0, \dots, p-1$ is invertible, we get that $f_i y^i = 0$; but y is invertible, so that $f_i = 0$. It follows that $\sum F[x] y^i$ has dimension p^2 over F, and is thus equal to A.

Now, from the asssumption it readily follows that $a = x^p - x$ and $b = y^p$ commute with x, y and are thus central, so x, y form a standard pair of generators. □

Now let

$$\mathrm{XY}_A = \{(x,y) \in \mathrm{X}_A \times \mathrm{Y}_A : \quad yxy^{-1} = x+1\}.$$

XY_A may by viewed as a bipartite graph, where the vertices are the elements of X_A and Y_A, and there is an edge between x and y iff $(x, y) \in \mathrm{XY}_A$. For an element to be in X_A or in Y_A depends on the characteristic polynomial, so we have $p-1$ (non linear) equations for each set. It follows that X_A and Y_A are (p^2-p+1)-dimensional varieties over F, and $\mathrm{XY}_A \subseteq \mathrm{X}_A \times \mathrm{Y}_A$ is a (p^2+1)-dimensional subvariety (as seen from Remark 2.3). In a sense, we study the geometry of XY_A.

Note that there are no isolated points on the graph:

Remark 2.2. (i) *For every $x \in \mathrm{X}_A$ there is some $y \in A$ such that $(x, y) \in \mathrm{XY}_A$.*
(ii) *Likewise for every $y \in \mathrm{Y}_A$, there is some $x \in A$ such that $(x, y) \in \mathrm{XY}_A$.*

Proof. (i) It is easy to see that $F[x]$ is either a subfield of dimension p of A, or isomorphic to the split ring $F^{\times p} = F \times \cdots \times F$. In both cases the automorphism induced by $x \mapsto x+1$ is inner (Skolem-Noether theorem, or the generalization to maximal separable commutative subalgebras in [3]), say induced by y. Then $F[x, y] = A$ be Remark 2.1.

(ii) This is [1, Theorem IV.17]. □

Two elements z, z' of $\mathrm{X}_A \cup \mathrm{Y}_A$ are said to be *at distance* $t/2$ if there is a chain of elements $z = z_0, z_1, \ldots, z_t = z' \in \mathrm{X}_A \cup \mathrm{Y}_A$ such that for every $i = 1, \ldots, t$, the couple z_{i-1}, z_i is a standard pair of generators. We take half of the usual distance in the graph XY_A, since we are sometimes more interested in the induced patterns on X_A or Y_A. We denote this situation by saying that $z \longleftrightarrow z_1 \longleftrightarrow \ldots \longleftrightarrow z_{t-1} \longleftrightarrow z'$ is a chain, where necessarily elements of X_A and Y_A interchange. We write X_A and Y_A in appropriate places in the chain to state existence of appropriate elements. For example, elements $x, x' \in \mathrm{X}_A$ are at distance 2 iff there is a chain $x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow \mathrm{Y}_A \longleftrightarrow x'$.

Let (x, y) be a standard pair of generators. The close neighborhood of x, y is described in the following remark.

Remark 2.3. (i) *The elements forming a standard pair of generators with x are of the form λy, where $\lambda \in F[x]^*$.*
(ii) *The elements forming a standard pair of generators with y are of the form $\mu + x$, where $\mu \in F[y]$.*

Proof. (i) $y_1 x y_1^{-1} = x + 1$ iff $y_1 y^{-1} \in C_A(F[x]) = F[x]$, and $y_1 y^{-1}$ is invertible since y, y_1 are.

(ii) $y x_1 y^{-1} = x_1 + 1$ iff $x_1 - x \in C_A(F[y]) = F[y]$. □

In particular, if $x \in \mathrm{X}_A$, then $x + \alpha \in \mathrm{X}_A$ for every $\alpha \in F$, and likewise for $y \in \mathrm{Y}_A$, $\beta y \in Y$ for every $\beta \in F^*$. We have

Remark 2.4. *The actions of F^+ and F^* on $\mathrm{X}_A, \mathrm{Y}_A$, respectively, define equivalence relations.*

In particular, if x, y are a standard pair of generators, $x' \equiv x$, and $y' \equiv y$, then x', y' are also a standard pair of generators.

The next proposition shows that there is essentially only one path connecting every two elements at distance 1.

Proposition 2.5. *Let $x, x' \in \mathrm{X}_A$ and $y, y' \in \mathrm{Y}_A$. If (x, y),(x, y'),(x', y) and (x', y') are all standard pairs of generators, then $x' \equiv x$ or $y' \equiv y$.*

Proof. By Remark 2.3, $\mu = x' - x \in F[y] \cap F[y']$, and $\lambda = y'y^{-1} \in F[x]$. Now $\lambda\mu\lambda^{-1} = \mu$, so that λ and μ commute. If A is a division ring, then we are done (as μ commutes with y, λ, so either $\mu \in F$ or $\lambda \in F$), but for the general case, write $\lambda = \sum \alpha_i x^i$ and $\mu = \sum \beta_j y^j$. Then compute $0 = [\mu, \lambda] = \sum \alpha_i \beta_j ((x + j)^i - x^i) y^j$, and compare the upper monomials with respect to y and x. We get a contradiction unless λ or μ are central. $\square$

The main result of this paper is the following

Theorem 2.6. *Let F be a field of characteristic $p = 3$, and let A be a (cyclic) division algebra of degree p over F.*

Then every two elements $x, z \in \mathrm{X}_A$ are at distance at most 3.

The proof is given in Section 4. This theorem can be reformulated in terms of presentations of algebras. Recall that for $a, b \in F$, $[a, b)_p$ denotes the p-algebra

$$[a, b)_p = F[x, y| \quad x^p - x = a, \quad y^p = b, \quad yxy^{-1} = x + 1].$$

Corollary 2.7. *Suppose $[a, b)_3 \cong [a', b')_3$ are two presentations of the same division algebra. Then there exist $a_1, a_2 \in F$ and $b_1, b_2, b_3 \in F^*$, such that*

$$[a, b) \cong [a, b_1) \cong [a_1, b_1) \cong [a_1, b_2) \cong [a_2, b_2) \cong [a_2, b_3) \cong [a', b_3) \cong [a', b').$$

One remark is in order concerning the split case. If $[a, b)_p \cong [a', b')_p$ are two presentations of $M_p(F)$, then

$$[a, b) \cong [0, b) \cong [0, b') \cong [a', b'),$$

so for a split algebra Corollary 2.7 holds, in a stronger form and for arbitrary p.

3. Elements at distance $1\frac{1}{2}$

Let A be a cyclic division p-algebra of degree p over F, where from now on we assume $p = \operatorname{char} F = 3$.

Fix a standard pair of generators $x, y \in A$, and set $\gamma = y^3 \in F$. In this section we classify the elements $u \in \mathrm{Y}_A$ which are at distance $1\frac{1}{2}$ from x, that is, elements for which there exists a chain

$$y \longleftrightarrow x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow u.$$

We denote by Tr the reduced trace map of A, and by tr the trace map of the extension $F[x]/F$. The action of y by conjugation on $F[x]$ is denoted by σ, and the notation $\mathrm{N}(\lambda)$ is preserved for the norm of elements in $F[x]$. Since $A = F[x, y] = \sum F[x]y^j$, we can write every $u \in A$ in the form $u = \lambda_0 + \lambda_1 y + \lambda_2 y^2$ for unique $\lambda_0, \lambda_1, \lambda_2 \in F[x]$. Set $\eta = \lambda_1 \cdot \sigma\lambda_2$.

Remark 3.1. *Assuming $u \notin F$, we have that $u \in \mathrm{Y}_A$ iff $\mathrm{Tr}(u) = \mathrm{Tr}(u^2) = 0$. As $\mathrm{Tr}(\lambda y) = \mathrm{Tr}(\lambda y^2) = 0$ for every $\lambda \in F[x]$, a simple computation yields the following equivalent conditions:*

$$\mathrm{tr}(\lambda_0) = 0 \tag{1}$$

$$\gamma\,\mathrm{tr}(\eta) = \mathrm{tr}(\lambda_0^2) \tag{2}$$

Under these assumptions, one can compute that $u^3 = \mathrm{N}(\lambda_0) + \gamma\,\mathrm{N}(\lambda_1) + \gamma^2\,\mathrm{N}(\lambda_2)$.

Lemma 3.2. *The element u is at distance $1\frac{1}{2}$ from x if and only if the following equations have a solution with $f_1, f_2 \in F$, $\lambda \in F[x]^*$:*

$$f_1\gamma(\lambda \cdot \sigma\lambda_2 - \sigma^2\lambda \cdot \lambda_2) + f_2\gamma \cdot \sigma\lambda \cdot (\lambda \cdot \sigma^2\lambda_1 - \sigma^2\lambda \cdot \lambda_1) = -\lambda_0 \tag{3}$$

$$f_1(\sigma\lambda_0 - \lambda_0) + f_2\gamma(\sigma\lambda \cdot \sigma^2\lambda_2 - \sigma^2\lambda \cdot \lambda_2) = 0 \tag{4}$$

$$f_1(\lambda \cdot \sigma\lambda_1 - \sigma\lambda \cdot \lambda_1) + f_2 \cdot \lambda \cdot \sigma\lambda \cdot (\sigma^2\lambda_0 - \lambda_0) = \lambda_2 \tag{5}$$

Proof. The elements x,u are at distance $1\frac{1}{2}$ iff there are some $y' \in \mathrm{Y}_A$ and $x' \in \mathrm{X}_A$ such that $x \longleftrightarrow y' \longleftrightarrow x' \longleftrightarrow u$ form a chain. By Lemma 2.3, we can write $y' = \lambda y$ for some $\lambda \in F[x]$, and then $x' - x \in F[\lambda y]$. Thus $x' = x + f_0 + f_1\lambda y + f_2(\lambda y)^2$ for some $f_0, f_1, f_2 \in F$, and by Remark 2.4 we may take $f_0 = 0$. The only remaining condition is that $ux' - x'u = u$, and comparing coefficients of y in both sides, we get Equations (3)–(5). □

Let $K = F[x]$ be a cyclic extension of dimension 3 of F, as before. The following facts are easily checked.

Remark 3.3. (i) *For every $\alpha_0, \alpha_1, \alpha_2 \in F$, we have that $\mathrm{tr}_{K/F}(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = -\alpha_2$.*

(ii) *For every* $\phi \in K$, *if* $\mathrm{tr}_{\mathrm{K/F}}\,\phi = \mathrm{tr}_{\mathrm{K/F}}\,\phi^2 = 0$, *then* $\phi \in F$.
(iii) *The map* $(\sigma-1) : K \to K$ *defined by* $(\sigma-1)a = \sigma(a) - a$ *is onto the subspace* $\{\phi \in K \colon \mathrm{tr}_{K/F}\,\phi = 0\}$.
(iv) $\mathrm{tr}_{K/F}\,\phi = 0$ *iff* $(\sigma-1)\phi \in F$.

Proof. (i) follows since the minimal polynomial of x is of the form $x^3 - x - \theta = 0$. (ii),(iii) follow trivially from (i), and (iv) follows since for every $\phi \in K$ we have that $(\sigma-1)^2\phi = (\sigma^2+\sigma+1)\phi$. □

Proposition 3.4. *Assume* $u = \lambda_0 + \lambda_1 y + \lambda_2 y^2$ *as before, and* $\lambda_0 \in F$. *Then* u *is at distance* $1\frac{1}{2}$ *from* x *if and only if the following holds:*
a. $\lambda_2 = 0$, **or**
b. $\lambda_2 \neq 0$, $\lambda_1 \neq 0$ *and* $\eta \notin F$, **or**
c. $\lambda_2 \neq 0$, $\lambda_1 \neq 0$, $\eta \in F$, *and* $\lambda_0\,\mathrm{N}(\lambda_1) = \eta^2\gamma$.

These conditions may look a little less random in light of the following observation: assuming $\lambda_0 \in F$, we have that $\eta \in F$ iff $F[u] = F[\lambda_1 y]$. If this is the case, then $u^2 \in F + F(\lambda_1 y)^2$ iff $\lambda_0\,\mathrm{N}(\lambda_1) = \eta^2\gamma$.

Proof. **Case 1:** $\lambda_2 = 0$. We must have $\lambda_1 \neq 0$, for otherwise $u = \lambda_0 \in F[x]$ would be separable. If $\lambda_0 = 0$, then by Remark 2.3.(i) we have the chain $y \longleftrightarrow x \longleftrightarrow y \longleftrightarrow x \longleftrightarrow \lambda_1 y = u$. Otherwise, choose $f_1 = 0$. Substituting, we find that Equations (4) and (5) are satisfied, and Equation (3) becomes

$$f_2\gamma\left(\sigma^2(\frac{\lambda_1}{\lambda}) - \frac{\lambda_1}{\lambda}\right) \cdot \mathrm{N}(\lambda) = -\lambda_0,$$

which can be solved by choosing $\lambda = x^{-1}\lambda_1$ and $f_2 = \frac{\theta\lambda_0}{\gamma\,\mathrm{N}(\lambda_1)}$, where $\theta = \mathrm{N}(x) \in F$. This results in the chain

$$x \longleftrightarrow y' = x^{-1}\lambda_1 y \longleftrightarrow x + \lambda_0 {y'}^{-1} \longleftrightarrow u.$$

Case 2: $\lambda_2 \neq 0$. If $\lambda_1 = 0$ then equation (5) has no solution. Thus we assume $\lambda_1 \neq 0$. In particular, $\eta = \lambda_1 \cdot \sigma\lambda_2 \neq 0$.

Case 2.1: $\eta \notin F$. Choose $f_2 = 0$ and $f_1 = 1$. Then Equation (4) vanishes, and substituting $\lambda_2 = \sigma^2\eta/\sigma^2\lambda_1$, Equations (3),(5) become

$$\begin{aligned} \eta\lambda/\lambda_1 - \sigma^2(\eta\lambda/\lambda_1) &= -\lambda_0/\gamma, \\ \lambda/\lambda_1 - \sigma(\lambda/\lambda_1) &= \sigma^2(\eta)/\mathrm{N}(\lambda_1), \end{aligned}$$

which is solved by $\lambda = \frac{\lambda_2 \cdot \sigma^2(\lambda_2) - \gamma^{-1}\lambda_0\lambda_1}{\sigma(\eta) - \eta}$. This satisfies $\lambda \neq 0$, for otherwise $\gamma\,\mathrm{N}(\lambda_2) = \lambda_0\eta$, contrary to the assumption $\eta \notin F$. Then we have the following chain: $x \longleftrightarrow \lambda y \longleftrightarrow x + \lambda y \longleftrightarrow u$.

Case 2.2: $\eta \in F$. We cannot have $f_2 \neq 0$, for then Equation (4) will force $\lambda/\lambda_1 \in F$, and from Equation (5) we then get $\eta = 0$, contrary to the assumption $\lambda_2 \neq 0$. Thus we have $f_2 = 0$, and the equations become

$$f_1\left(\sigma(\frac{\lambda}{\lambda_1}) - \frac{\lambda}{\lambda_1}\right) = \frac{-\lambda_0}{\gamma\eta} = \frac{-\sigma^2(\eta)}{\mathrm{N}(\lambda_1)},$$

for which, by Remark 3.3.(iii), there is a solution λ iff $\lambda_0 \,\mathrm{N}(\lambda_1) = \gamma\eta^2$. Indeed we can take $f_1 = 1$ and $\lambda = -\lambda_2\sigma(\lambda_1)^{-1}x$, and the resulting chain is $x \longleftrightarrow \lambda y \longleftrightarrow x + \lambda y \longleftrightarrow u$ □

Corollary 3.5. *Let $x \in \mathrm{X}_A$, then x and $-x$ are at distance at least 3.*

Proof. Choose y such that $(x, y) \in \mathrm{XY}_A$. We show that there is no chain $y \longleftrightarrow x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow u \longleftrightarrow -x \longleftrightarrow y^2$. Every appropriate u is, by Remark 2.3.(i), of the form $u = \lambda y^2$ for some $\lambda \in F[-x] = F[x]$, and then the completion is impossible by the last proposition. □

Corollary 3.6. *For every $y \in \mathrm{Y}_A$, the distance between y and y^2 is at least 3.*

Proof. Otherwise, there is a chain

$$y \longleftrightarrow x' \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow y^2,$$

but since $-x', y^2$ form a standard pair of generators, it follows that the distance between x' and $-x'$ is at most 2, contrary to the former corollary. □

For the rest of the section we no longer assume $\lambda_0 \in F$. Let $b = \sigma(\lambda_0) - \lambda_0$, then $b \in F$ by Equation (1) and Remark 3.3.(iv). Moreover, since $\mathrm{tr}(\lambda_0) = 0$, we have that $\lambda_0 = a + bx$ for $a \in F$.

Proposition 3.7. *Let x, y form a standard pair of generators and $u = \lambda_0 + \lambda_1 y + \lambda_2 y^2 \in \mathrm{Y}_A$ where $\lambda_0 = a + bx$ and $\eta = \lambda_1 \cdot \sigma\lambda_2$ as above. If $\lambda_0 \notin F$ and $\gamma(\sigma^2\eta - \eta) = b\lambda_0$, then u is at distance $1\frac{1}{2}$ from x.*

Proof. Set $x' = x - b^{-1}\lambda_2 y^2$ and $y' = \lambda_2 \cdot \sigma^2\lambda_2 \cdot y$. Then the first two pairs in the chain

$$x \longleftrightarrow y' \longleftrightarrow x' \longleftrightarrow u$$

follow from Remark 2.3. For the third pair, compute that $ux' - x'u = \gamma b^{-1}(\sigma^2\eta - \eta) + \lambda_1 y + \lambda_2 y^2$, which equals u by the assumption. □

Note that the assumption $\gamma(\sigma^2\eta - \eta) = b\lambda_0$ implies (but is not implied by) Equation (2).

The following remark is given as a counterpart for Proposition 3.4, and is not needed later.

Remark 3.8. *Assume $u = \lambda_0 + \lambda_1 y + \lambda_2 y^2$ as before, and $\lambda_0 \notin F$. Then there exist homogeneous quadratic forms $\mathcal{Q}_I, \mathcal{Q}_{II}$ in two variables over F, explicitly stated in the proof, such that u is at distance $1\frac{1}{2}$ from x if and only there are $g, f_1 \in F$ such that*

$$\begin{aligned}\mathcal{Q}_I(g, f_1) &= 0,\\ \mathcal{Q}_{II}(g, f_1) &\neq 0.\end{aligned}$$

Proof. Since $\lambda_0 \notin F$, by Remark 3.3.(ii) we have that $\operatorname{tr}\lambda_0^2 \neq 0$, so by Equation (2), $\eta \neq 0$ and thus also $\lambda_1, \lambda_2 \neq 0$. Moreover, Equation (4) has no solution with $f_2 = 0$, so we may assume $f_2 \neq 0$.

We write

$$\lambda = \frac{g + f_1 \cdot \sigma^2\lambda_0}{f_2\gamma \cdot \sigma\lambda_2} \tag{6}$$

for $g \in K$. Then Equation (4) is equivalent to $g \in F$, and we assume this is the case. Recall that by Lemma 3.2 we need to solve Equations (3)–(5) with $f_1, f_2 \in F$ and $\lambda \in K$, so this now becomes solving Equations (3) and (5) with $f_1, f_2, g \in F$, $f_2 \neq 0$. Write $\lambda_0 = a + bx$ with $a, b \in F$. Note that from Equation (2) and Remark 3.3.(i), we get that $\gamma\operatorname{tr}(\eta) = \operatorname{tr}(\lambda_0^2) = -b^2$.

Denote by

$$\begin{aligned}\mathcal{Q}_0(s,t) = (\sigma^2\eta - \eta)s^2 &+ (\sigma^2\lambda_0 \cdot \eta - \sigma\lambda_0 \cdot \sigma^2\eta)st\\ &+ (\gamma b\,\mathrm{N}(\lambda_2) + \lambda_0(\sigma^2\lambda_0 \cdot \sigma^2\eta - \sigma\lambda_0 \cdot \eta))t^2,\\ \mathcal{Q}_2(s,t) = -bs^2 &+ (\gamma(\sigma\eta - \eta) + b \cdot \sigma\lambda_0)st\\ &+ (\gamma(\sigma^2\lambda_0 \cdot \sigma\eta - \lambda_0\eta) - b\lambda_0 \cdot \sigma^2\lambda_0)t^2\end{aligned}$$

the two quadratic forms in s, t over K.

Substituting (6) in Equations (3),(5), multiplying by $f_2\gamma\,\mathrm{N}(\lambda_2)$ in the first case and by $f_2\gamma^2 \cdot \sigma\lambda_2 \cdot \sigma^2\lambda_2$ in the second case, we get the following system of equations, in the variables $g, f_1 \in F$, $f_2 \in F^*$:

$$\mathcal{Q}_0(g, f_1) = -\lambda_0 f_2\gamma\,\mathrm{N}(\lambda_2) \tag{7}$$

$$\mathcal{Q}_2(g, f_1) = f_2\gamma^2\,\mathrm{N}(\lambda_2). \tag{8}$$

It can be checked that $\mathcal{Q}_2$ is actually a quadratic form over F. $\mathcal{Q}_0$, however, is not defined over F (the coefficient $\sigma^2\eta - \eta \notin F$, for otherwise we would have $b^2 = \gamma\operatorname{tr}\eta = 0$ by Remark 3.3.(iv)).

Fortunately we have that $\operatorname{tr}\mathcal{Q}_0 = 0$, so by Remark 3.3.(i), the coefficients of $\mathcal{Q}_0$ lay in the two dimensional F-space $F + F\lambda_0 \subset K$. Write

$$\mathcal{Q}_0 = \mathcal{Q}_I + \lambda_0\mathcal{Q}_{II}$$

for the respective components. Then we can compute $\mathcal{Q}_{II} = \frac{1}{b}(\sigma(\mathcal{Q}_0) - \mathcal{Q}_0)$ to be

$$\begin{aligned}\mathcal{Q}_{II}(s,t) = -\operatorname{tr}(\eta)/b \cdot s^2 \quad &+ \quad \operatorname{tr}(\lambda_0 \cdot \sigma\eta)/b \cdot st \\ &- \quad \operatorname{tr}(\lambda_0 \cdot \eta \cdot \sigma\lambda_0)/b \cdot t^2\end{aligned}$$

and so $\mathcal{Q}_I = \mathcal{Q}_0 - \lambda_0 \mathcal{Q}_{II}$ is

$$\begin{aligned}\mathcal{Q}_I(s,t) = (\sigma^2\eta - \eta - b\lambda_0/\gamma)s^2 \quad &+ \quad (\sigma^2\lambda_0 \cdot \sigma^2\eta - \sigma\lambda_0 \cdot \eta + b\lambda_0^2/\gamma)st \\ &+ \quad (\gamma b \operatorname{N}(\lambda_2) - b \operatorname{N}(\lambda_0)/\gamma)t^2.\end{aligned}$$

It may not be so obvious, but one can check that $\mathcal{Q}_I$ is indeed defined over F.

Using this decomposition, Equation (7) now becomes

$$\mathcal{Q}_I(g, f_1) = 0 \tag{9}$$

$$\mathcal{Q}_{II}(g, f_1) = -f_2\gamma \operatorname{N}(\lambda_2) \tag{10}$$

Again this is not immediate, but one can compute that $\mathcal{Q}_2 = -\gamma\mathcal{Q}_{II}$. Thus solving Equations (7),(8) is equivalent to solving Equations (9) and (10). Recall that we only assumed $f_2 \neq 0$, so all we have to do is find a zero of $\mathcal{Q}_I$ which is not a zero of $\mathcal{Q}_{II}$, as claimed. □

Example 3.9. *Suppose $\gamma(\sigma^2\eta - \eta) = b\lambda_0$ as in Proposition 3.7. The coefficient $-tr(\eta)/b = b/\gamma$ of s^2 in the form $\mathcal{Q}_{II}(s,t)$ is nonzero, so if we substitute $f_1 = 0$ and $g = 1$ in $\mathcal{Q}_I, \mathcal{Q}_{II}$ we get $\mathcal{Q}_I(1,0) = \sigma^2\eta - \eta - b\lambda_0/\gamma = 0$ and $\mathcal{Q}_{II}(1,0) = -\operatorname{tr}(\eta)/b \neq 0$. By Remark 3.8, u is at distance $1\frac{1}{2}$ from x, in accordance with the above mentioned proposition.*

4. A Proof of Theorem 2.6

Let A be a division p-algebra of degree $p = 3$. We are given two elements $x, z \in \mathrm{X}_A$, and wish to find a chain

$$x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow \mathrm{Y}_A \longleftrightarrow z.$$

Choose (using Remark 2.2.(i)) elements y, u, such that x, y and z, u are standard pairs of generators. For x, y, u we use the notations of the previous section: σ is the action of conjugation by y on $F[x]$, $\operatorname{N}(\lambda)$ is preserved for the norm of elements in $F[x]$, $u = \lambda_0 + \lambda_1 y + \lambda_2 y^2$ for $\lambda_0, \lambda_1, \lambda_2 \in F[x]$, and $\eta = \lambda_1 \cdot \sigma\lambda_2$. Also $b = \sigma\lambda_0 - \lambda_0$, and $\lambda_0 = a + bx$ for $a, b \in F$. Similarly, whenever we specify an element u', the same notation is used: $u' = \lambda_0' + \lambda_1' y + \lambda_2' y^2$, $\eta' = \lambda_1' \cdot \sigma\lambda_2'$, and $\lambda_0' = a' + b'x$.

Remark 4.1. *For every $\alpha \in F$ we have that $u + \alpha$ is at distance $1\frac{1}{2}$ from z.*

Proof. Case 1 of Proposition 3.4 (with z, u in place of x, y and $u+\alpha$ in place of u) gives the chain

$$u+\alpha \longleftrightarrow z+\alpha u^{-1}z \longleftrightarrow z^{-1}u \longleftrightarrow z \longleftrightarrow u.$$

□

Proof of Theorem 2.6. **Case 1:** $\lambda_0 \in F$. Note that $\operatorname{tr}(\eta) = 0$ by Equation (2). If $\lambda_2 = 0$, then we have the chain

$$y \longleftrightarrow x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow u \longleftrightarrow z$$

by Proposition 3.4. So we assume $\lambda_2 \neq 0$.

Case 1.1: $\lambda_1 = 0$. Then $u = \lambda_0 + \lambda_2 y^2$. Set $\tilde{z} = -x - \frac{\lambda_0 x}{\gamma\cdot\sigma\lambda_2}y$, and check that $\tilde{z}, u$ form a standard pair of generators. Compute that $\tilde{z}u = \lambda_0 x - \frac{\lambda_0^2 x}{\gamma\cdot\sigma\lambda_2}y - x\lambda_2 y^2$, and set $u' = \tilde{z}u$. Then for u' we have $b' = \sigma(\lambda_0 x) - \lambda_0 x = \lambda_0$, $a' = \lambda_0' - b'x = 0$, and $\eta' = \lambda_1' \cdot \sigma\lambda_2' = \frac{1}{\gamma}\lambda_0^2(x+x^2)$.

Case 1.1.1: $\lambda_0 \neq 0$ (so that $\lambda_0' \notin F$). Compute that $\gamma(\sigma^2\eta' - \eta') = x\lambda_0^2 = b'\lambda_0'$, so by Proposition 3.7, u' is at distance $1\frac{1}{2}$ from x, and we have the following chain:

$$y \longleftrightarrow x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow u' \longleftrightarrow \tilde{z} \longleftrightarrow u \longleftrightarrow z.$$

Case 1.1.2: $\lambda_0 = 0$. Then $u = \lambda_2 y^2$ and thus $-x+u, u$ form a standard pair of generators. Choose $u' = (-x+u)u = (\gamma\lambda_2 \cdot \sigma^2\lambda_2)y - (x\lambda_2)y^2$, so that we have $\lambda_0' = 0$, $\lambda_2' \neq 0$, $\lambda_1' \neq 0$, and $\eta' = \lambda_1' \cdot \sigma\lambda_2' = -\gamma\,\mathrm{N}(\lambda_2)\sigma(x) \notin F$. By Case b. of Proposition 3.4, u' is at distance $1\frac{1}{2}$ from x, and the resulting chain is

$$y \longleftrightarrow x \longleftrightarrow \lambda y \longleftrightarrow x+\lambda y \longleftrightarrow u' \longleftrightarrow -x+\lambda_2 y^2 \longleftrightarrow \lambda_2 y^2 = u \longleftrightarrow z$$

for $\lambda = -\frac{x(x-1)}{\gamma\cdot\sigma\lambda_2}$.

Case 1.2: $\lambda_1 \neq 0$. If $\lambda_2 = 0$, or $\lambda_2 \neq 0$ but $\eta \notin F$, there is a chain

$$y \longleftrightarrow x \longleftrightarrow \mathrm{Y}_A \longleftrightarrow \mathrm{X}_A \longleftrightarrow u \longleftrightarrow z$$

by Proposition 3.4. So suppose $\lambda_2 \neq 0$, and $\eta \in F^*$. Choose $\alpha = \frac{\gamma\eta^2}{\mathrm{N}(\lambda_1)} - \lambda_0$, then for $u' = u+\alpha$ we have that $\lambda_1' = \lambda_1 \neq 0$, $\lambda_2' = \lambda_2 \neq 0$, $\lambda_0' = \alpha + \lambda_0 \in F$, and $\eta' = \eta$. But now we have $\lambda_0'\,\mathrm{N}(\lambda_1') = \eta'^2\gamma$, so from Case c. of Proposition 3.4 and Remark 4.1, we get the chain

$$y \longleftrightarrow x \longleftrightarrow \lambda y \longleftrightarrow x+\lambda y \longleftrightarrow u' \longleftrightarrow z_\alpha u^{-1}z \longleftrightarrow z^{-1}u \longleftrightarrow z \longleftrightarrow u$$

where $\lambda = -\frac{x\lambda_2}{\sigma\lambda_1}$.

Case 2: $\lambda_0 \notin F$. In view of the Remark 4.1, it is enough to show that there is some $\alpha \in F$ such that $x, u+\alpha$ are at distance $1\frac{1}{2}$. Recall that $\lambda_0 = bx + a$ where $a, b \in F$, so by Equation (2) we also have $\gamma\eta = \eta_0 + \eta_1 x + b^2x^2$ for $\eta_0, \eta_1 \in F$. Choose $\alpha = b - a - \eta_1/b$, then for

$u' = u + \alpha$ we have that $\eta' = \eta$, and $\gamma(\sigma^2\eta - \eta) = b^2(x+1) - \eta_1 = b\lambda_0 + b\alpha = b\lambda_0'$. By Proposition 3.7 we thus have the chain

$$x \longleftrightarrow \lambda_2 \cdot \sigma^2\lambda_2 \cdot y \longleftrightarrow x - b^{-1}\lambda_2 y^2 \longleftrightarrow u' \longleftrightarrow z + \alpha u^{-1} z \longleftrightarrow z^{-1} u \longleftrightarrow z,$$

and we are done. □

5. The geometry of XY_A

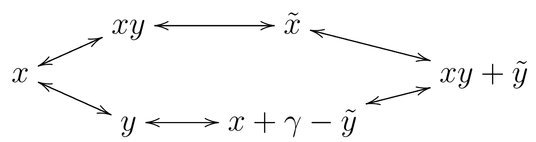
Let A be a division algebra of degree 3 over a field F of characteristic $p = 3$. In this section we describe some properties of the graph XY_A and the graphs induced on X_A and Y_A, and present some special subgraphs. It seems reasonable to slightly alter the notation for this purpose.

Recall the equivalence relations defined in Remark 2.4. In this section we let X_A,Y_A denote the sets of equivalence classes (rather than the sets of points, as done previously). Again, XY_A is the bipartite graph whose vertices are $\mathrm{X}_A \cup \mathrm{Y}_A$, with an edge connecting the classes $[x], [y]$ iff x, y are a standard pair of generators. We view X_A and Y_A as subgraphs of XY_A, where two points $x, x' \in \mathrm{X}_A$ are connected iff there is there is some $y \in \mathrm{Y}_A$ such that $(x, y), (x', y) \in \mathrm{XY}_A$. Thus the distance induced by XY_A on X_A and Y_A is the usual distance in graphs.

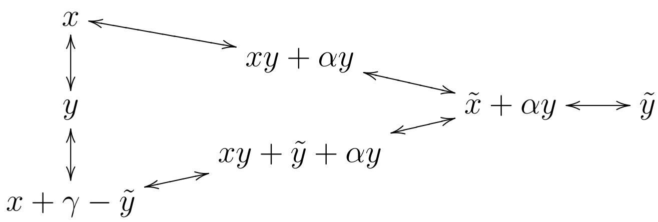
Theorem 2.6 bounds the diameter of X_A to be ≤ 3, and this bound is shown to be exact in Corollary 3.5. Applying Remark 2.2, we see that the diameter of Y_A is bounded by 4. A lower bound of 3 is given by Corollary 3.6

Fix some $y \in \mathrm{Y}_A$. The elements $x \in \mathrm{X}_A$ connected to y are at distance 1 from one another, so they form a complete subgraph in X_A. The same thing happens in Y_A around any $x \in \mathrm{X}_A$.

Subgraphs of XY_A are more interesting. Proposition 2.5 shows that X_A and Y_A are simple graphs (i.e., there are no multiple paths between neighbors). It follows that XY_A does not contain squares. Let x, y be a fixed standard pair of generators, and let $\gamma = y^3$. Set $\tilde{y} = \gamma + y - y^2$, and $\tilde{x} = x + xy$. Then we get the following hexagon:



This can be generalized, to the following:



For every $\alpha \in F$, this figure is a triangle in X_A, together with the corresponding triangle in Y_A. As α varies, the complex is rotated along the fixed axis $x \longleftrightarrow y \longleftrightarrow x + \gamma - \tilde{y}$, but with all the heads of the resulting triangles connected to a single point $\tilde{y}$. In particular, we get infinitely many different chains of length $1\frac{1}{2}$ connecting x and $\tilde{y}$. It also shows a point (x) connected to a star (the points $\{\tilde{x} + ay\}$ around $\tilde{y}$) but not to its center, and other similar phenomenon.

References

[1] A. A. Albert, Structure of Algebras, Amer. Math. Soc. Coll. Publ., Vol. XXIV, Providence, 1961.

[2] S. Amitsur, L. H. Rowen and J.-P. Tignol, *Division Algebras of Degree* 4 *and* 8 *with Involution*, Israel J. Math. **33**(2), 133–148, (1979).

[3] S. U. Chase, *Two Results on Central Simple Algebras*, Comm. Algebra **12**, 2279–2289, (1984).

[4] W. Fischer, On Twisted Cyclic Algebras and the Chain Equivalence of Kummer Elements, Doctoral Dissertation, Regensburg Univ., Germany, July 1999.

[5] N. Jacobson, Finite Dimensional Division Algebras over Fields, Springer, 1996.

[6] M. Rost, *The chain lemma for Kummer elements of degree* 3, C. R. Acad. Sci. Paris, **t. 328**, Série I, 185-190, (1999).

[7] L. H. Rowen, Ring Theory, Academic Press, 1988.

[8] U. Vishne, Central Simple Algebras, Doctoral Dissertation, Bar-Ilan Univ., Israel, July 2000.

E-mail address: `vishne@math.huji.ac.il`