Available online at www.sciencedirect.com
ScienceDirect
Advances in Applied Mathematics 37 (2006) 511–513
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

An affine prime non-semiprimitive monomial algebra with quadratic growth

Agata Smoktunowicz [1], Uzi Vishne [*,2]

Department of Mathematics, Yale University, 10 Hillhouse Avenue, New-Haven, CT 06520, USA

Received 19 January 2005; accepted 9 March 2005

Available online 2 May 2006

Dedicated to Amitai Regev on his 65th birthday

Abstract

We give a simple construction of a prime monomial algebra with quadratic growth, which is neither primitive nor PI.
© 2006 Elsevier Inc. All rights reserved.

In [5], Markov showed how to embed a countably generated algebra as a right ideal of an affine algebra. His ideas were later extended by Beidar [2] and L. Small (see [6, Example 6.2.3] and [4, Subsection 9.2.2]), who suggested the term 'affinization' for this process. In [3], Bell uses affinization to construct various algebras with interesting properties and small Gelfand–Kirillov dimension. In particular he answers (negatively) an old question of Small, who asked if a prime affine algebra of Gelfand–Kirillov dimension 2 is necessarily primitive or PI.

Recall that the Gelfand–Kirillov dimension of an affine algebra R is defined as

$$\mathrm{GK}\dim(R) = \limsup \frac{\dim(V^n)}{\log(n)}$$

* Corresponding author.
E-mail addresses: A.Smoktunowicz@ed.ac.uk (A. Smoktunowicz), vishne@math.biu.ac.il (U. Vishne).
1 Current address: School of Mathematics, University of Edinburgh, JCMB, Kings' Buildings, Mayfield Road EH 9 3JZ Edinburgh, Scotland, UK.
2 Current address: Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel.

0196-8858/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.aam.2005.03.008

where V is a generating subspace (containing the identity). For a non-affine algebra, $\mathrm{GK}\dim(R)$ is defined as the supremum of $\mathrm{GK}\dim(R_0)$ over affine subalgebras R_0. The key idea in affinization is to cover a countably generated prime algebra T by the 'corner' $e_{11}Ae_{11}$ of a certain affine matrix algebra A, and then to use Zorn's lemma to obtain a quotient algebra of A whose corner is equal to T. Bell shows how to compute the Gelfand–Kirillov dimension of the affinization of T in terms of the dimension of T itself. However, the non-constructiveness of Zorn's lemma and the fact that T is not affine to begin with, make it difficult to control the precise growth function of the resulting algebra. Indeed, in [3, Question 3.2] Bell repeats Small's question for algebras of quadratic growth.

Recently, Bartholdi [1] showed that an affine 'recurrent transitive' algebra (without unit) constructed from Grigorchuk's group of intermediate growth, is prime and of quadratic growth; moreover, assuming the base field is an algebraic extension of $\mathbb{F}_2$, the algebra is Jacobson radical and not nil.

In [7] Zelmanov constructed an affine prime monomial algebra with a non-zero locally nilpotent ideal. In this note we adjust his example to obtain a prime, monomial algebra R with quadratic growth (over arbitrary base field), which is not semiprimitive. Thus R is not primitive, and also does not satisfy a polynomial identity (for otherwise $\mathrm{Jac}(R)$ would be nilpotent by the Razmyslov–Kemer–Braun theorem, which is impossible since R is prime).

Let $\{t_n\}$ be a weakly increasing unbounded sequence of integers. Define words in the free monoid $\langle x, y\rangle$ by

$$v_1 = x,$$
$$v_{n+1} = v_n y^{t_n} v_n.$$

The limit v_∞ is a well defined infinite sequence of the letters x, y, since every v_n is a header in v_{n+1}.

Let k be an arbitrary field, and set R to be the free associative algebra $k\langle x, y\rangle$ modulo the ideal generated by monomials which are not subwords of v_∞. In comparison, in Zelmanov's example the number of xs in a non-zero monomial is bounded by a function of the length.

R is a prime algebra since if $u, u' \leqslant v_\infty$ are non-zero monomials then $u, u' \leqslant v_n$ for some n, and then $uwu' \leqslant v_{n+1}$ for an appropriate monomial w.

Proposition 1. *The ideal $\langle x\rangle$ of R is locally nilpotent.*

Proof. Let $a_1, \ldots, a_s \in \langle x\rangle$, and let m be the length of the longest monomial of the a_i. Choose n such that $m < t_n$. Let u be a non-zero monomial in a product of the a_i. As a product of monomials of the a_i, w must have an x every at most m letters. Every subword of v_∞ is in the monoid generated by v_n and powers y^t for $t \geqslant t_n > m$, so w must be a subword of $y^m v_n y^m$. This shows that any product of more than $|y^m v_n y^m|$ of the a_i is zero. $\square$

It follows that $\langle x\rangle \subseteq L(R) \subseteq \mathrm{Jac}(R)$ (where $L(R)$ is the Levitzki radical, namely the largest locally nilpotent two-sided ideal). To prove the claims in the title, it remains to bound the growth rate:

Proposition 2. *If $|v_n|/t_n$ is bounded, then R has quadratic growth.*

Proof. Let $V = kx + ky$. To show that $\dim((k+V)^m)$ is quadratic, we need to show that $\dim(V^m)$ is linear in m. By definition of R, $\dim(V^m)$ is the number of subwords of length m of the v_i. Choosing n so that $m < |v_n|$ and $m < t_n$, we have that $y^m v_n \leqslant y^{t_n} v_n \leqslant v_{n+1}$ and since v_n begins with an x, v_{n+1} has at least m different subwords of length m. Thus $\dim(V^m) \geqslant m$.

Now let c be such that $|v_n|/t_n < c$, and choose n such that $t_n < m \leqslant t_{n+1}$. Then the segments of length m in v_∞ are all segments in words of the form $y \cdots y v_{n+1} y \cdots y$, so there are no more than $m + |v_{n+1}| = m + 2|v_n| + t_n < m + (2c+1)t_n < (2c+2)m$ such segments. □

To obtain an explicit example, reminiscent of the Cantor set, choose $t_n = 3^{n-1}$. Then $|v_n| = 3^n$ and $|v_n|/t_n$ is a constant.

Remark 3. (a) R is not left-Noetherian (since the left ideal $\sum R y^{t_n} x$ is not finitely generated).

(b) Arbitrarily long sequences of ys are non-zero and so the subalgebra generated by y is isomorphic to the ring of polynomials in y. Moreover $R/\langle x \rangle \cong k[y]$, and since this is a semi-primitive ring, $\mathrm{Jac}(R) = \langle x \rangle$.

Question 4. Is every prime semiprimitive algebra with Gelfand–Kirillov dimension 2 (or: quadratic growth) necessarily primitive or PI?

References

[1] L. Bartholdi, Branch rings, thinned rings, tree enveloping rings, Israel J. Math., in press.

[2] K.I. Beĭdar, Radicals of finitely generated algebras, Uspekhi Mat. Nauk 36 (6) (1981) 203–204.

[3] J. Bell, Examples in finite Gelfand–Kirillov dimension, J. Algebra 263 (2003) 159–175.

[4] A.A. Belov, L.H. Rowen, Polynomial Identities: A Combinatorial Approach, Academic Press, 2005.

[5] V.T. Markov, Some examples of finitely generated algebras, Russ. Math. Surv. 36 (5) (1981) 167–168. MR0637450.

[6] L.H. Rowen, Ring Theory, Pure and Appl. Math., vol. 127, Academic Press, 1988.

[7] E. Zelmanov, An example of a finitely generated prime ring, Sib. Math. J. 20 (1979) 303–304.