NONCOMMUTATIVE INCLUSION-EXCLUSION, REPRESENTATIONS OF LEFT REGULAR BANDS AND THE TSETLIN LIBRARY

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ABSTRACT. We find a semigroup Q_n, whose category of partial representations contains the representation category $\mathbf{Rep}(\mathcal{F}_n)$ of the free left regular band $\mathcal{F}_n$. We use this to construct a resolution for the absolute kernel of a representation of $\mathcal{F}_n$, for which the kernel Sp_n of the Markov operation in the Tsetlin library model is a prominent example. We obtain a formula for the dimension of the absolute kernel, generalizing the equality of the dimension of Sp_n to the number of derangements of order n.

1. Introduction

Counting derangements (namely permutations without fixed points) is a compulsory exercise on the inclusion-exclusion formula. This number is naturally encountered in a classical Markov model called the Tsetlin library: The state space of the Tsetlin library is the set S_n of permutations on n books placed on a single shelf; in each step, one of the books is selected (at random) and re-stacked at the end of the shelf. The n possible moves, $\epsilon_1, \dots, \epsilon_n$, generate a semigroup $\mathcal{F}_n$, which is the free "left regular band" on n generators. Viewing $F[S_n]$ as a linear representation of $\mathcal{F}_n$, where F is a field, the dimension of the "absolute kernel", $K = \bigcap \mathrm{Ker}(\epsilon_j)$, is equal to the number of derangements, which is $[e^{-1}n!]$.

This paper studies the absolute kernel of an arbitrary representation of $\mathcal{F}_n$. The key idea is to factor the generators ϵ_j as $\epsilon_j = \mu_j \pi_j$, where π_j and μ_j are partial linear transformations on a larger space, so that π_j and μ_j are weak inverses of each other, and all the π_j commute. This setup defines an ambient semigroup $Q_n \supset \mathcal{F}_n$, whose partial representations correspond to representations of $\mathcal{F}_n$, with $\mathrm{Ker}(\epsilon_j) = \mathrm{Ker}(\pi_j)$. This approach lets us explicitly construct a resolution (11) for K, based on inclusion-exclusion operators, cf. Theorem 5.3.

Motivated by [2], let us spell out an application in the language of noncommutative polynomials, which uses and extends exactness of (11) at $\mathbf{P}_{n-1}$. Fix $k < n$. For each subset $S \subseteq \{1, \dots, n\}$ of cardinality k, let f_S be a multilinear polynomial in the variables $\{x_i \,|\, i \in S\} \cup \{y_1, y_2, \dots\}$. We say that this system $\mathcal{S} = \{f_S : |S| = k\}$ is **coherent** if for every S of cardinality $k-1$, and every $i, i' \notin S$, we have that

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$(f_{S\cup\{i\}})|_{x_i=1} = (f_{S\cup\{i'\}})|_{x_{i'}=1}$. We say that a multilinear polynomial g in the variables $\{x_1,\dots,x_n\}\cup\{y_1,y_2,\dots\}$ is a **common ancestor** of $\mathcal{S}=\{f_S\}$ if for every subset S of cardinality k, $g|_{x_i\mapsto 1, i\notin S} = f_S$.

Theorem (Corollary 12.1)**.** *Every coherent system of multilinear polynomials has a common ancestor.*

Having a resolution at hand leads to a general formula for the dimension of the absolute kernel:

Theorem (Theorem 13.4)**.** *Let V be a finite dimensional representation space of $\mathcal{F}_n$, whose action is generated by endomorphisms $\epsilon_j \in \mathrm{End}(V)$. The absolute kernel has dimension*

$$\dim(\bigcap_j \mathrm{Ker}(\epsilon_j)) = \sum_{k=1}^{n}(-1)^{k-1}\sum_{i_1<\dots<i_k}\dim(\mathrm{Ker}(\epsilon_{i_1}\cdots\epsilon_{i_k})).$$

This paper has three parts. In Section 2 we define the semigroup Q_n and show that it contains the free regular band $\mathcal{F}_n$. In Section 3 we define inclusion-exclusion systems, which are partial representations of Q_n, and show that they induce representations of $\mathcal{F}_n$. In Section 4 we prove that every representation of $\mathcal{F}_n$ corresponds to a unique inclusion-exclusion system, establishing (in Theorem 4.6) the equivalence of categories $\mathbf{Rep}(\mathcal{F}_n)\cong\mathbf{IE}(Q_n)$.

Inclusion-exclusion systems are used in the second part to construct a resolution for the absolute kernel K of a representation of $\mathcal{F}_n$, namely the subspace $\bigcap_{e\neq 1}\mathrm{Ker}(e)$. For the Tsetlin library, this kernel can be viewed as the space of Spechtian polynomials. The resolution is described in Section 5, and we prove that it is indeed a resolution in Sections 6–8 by proving a relation of the form $\partial_k = \partial_k \circ s_k - s_{k-1}\circ\partial_{k-1}$, where s_k is the multilayered Tseltin library operator.

The final part provides some applications. We specialize to the case $n=2$ in Section 9 and provide explicit formulas. This is generalized in Section 10, where we show that coherence is always a result of a common ancestor. This is used to prove in Section 11 the exactness at $\mathbf{P}_{n-1}$ by an explicit, characteristic free, inclusion-exclusion formula. Further applications are given in Section 12, where we characterize systems of multilinear polynomials, on all subsets of a given variable set, that are obtained by substitution from a common polynomial. Diverging somewhat from the main theme of this paper, we also discuss distributions on partial permutations which are induced by a distribution on permutations of the full index set. Finally in Section 13 we derive, from the resolution, dimension formulas for the absolute kernel of any representation of $\mathcal{F}_n$.

2. An ambient semigroup for the free left regular band

Left regular bands are an important class of semigroups (see [3]). Of particular interest for us is the free left regular band $\mathcal{F}_n$, which can be used to analyze a famous

Markov chain model called the Tsetlin library [9, Section 15.3.1]. We construct a semigroup Q_n containing $\mathcal{F}_n$, and use it to find the eigenvalues for the Markov model.

2.1. **The free left regular band.** A **band** (also called an idempotent semigroup) is a semigroup whose elements are all idempotents (so that $u^2 = u$ for every u). A band satisfying the identity $uvu = uv$ is called a **left regular band**. Being defined by identities, the collection of left regular bands is a variety. The free left regular band on n generators $e_1, \dots, e_n$ is composed of the products $e_{i_1} \cdots e_{i_t}$, for distinct indices $i_1, \dots, i_t$, and can be defined by the relations

$$e_{i_1} \cdots e_{i_t} e_{i_1} = e_{i_1} \cdots e_{i_t} \qquad \text{distinct } i_1, \dots, i_t, \tag{1}$$

see for example [7, Example 2.2]. We denote this semigroup by $\mathcal{F}_n$.

Left regular bands appear in various contexts. For example, the faces in a hyperplane arrangement in the Euclidean space form a left regular band with respect to Tits projection operation (see [6]). The irreducible representations of finite bands over any field are 1-dimensional [3, Thm 3.1]. However, our main motivation is a natural $n!$-dimensional representation, which we generalize and describe in the language of noncommutative polynomials.

Example 2.1. *Fixing an arbitrary set Y, let $W_{n,Y}$ be the space of multilinear polynomials in the non-commuting variables $\{x_1, \dots, x_n\} \cup Y$. The semigroup $\mathcal{F}_n$ acts on $W_{n,Y}$ by*

$$e_i : f(x_1, \dots, x_n; Y) \mapsto f(x_1, \dots, 1, \dots, x_n; Y) x_i.$$

This representation (for $Y = \emptyset$) describes the "Tsetlin library", whose "books" are the letters $x_1, \dots, x_n$. The word $x_{\sigma(1)} \cdots x_{\sigma(n)}$ corresponds to a permutation $\sigma \in S_n$ of the books on a single shelf. The generator e_i can be viewed as restacking the book x_i at the end of the line. Applying the e_i at random (via some fixed distribution, reflecting the popularity of the books) defines a Markov chain on the $n!$ permutations. Taking a linear perspective, $\mathcal{F}_n$ acts on the space $W_{n,\emptyset}$. In particular it acts on its subset of **distribution elements**, namely elements whose coefficients sum up to 1, as they encode (over the real numbers when the coefficients are positive) distributions on the $n!$ permutations. Likewise it acts on the space of elements whose coefficients sum up to zero, which is of interest in PI-theory. The eigenvalues and eigenstates of this action on $W_{n,\emptyset}$ are known, [9, Section 15.6.1].

2.2. **An ambient semigroup.** What if we borrow a book and do not immediately restack it? The library will not hold the complete set of books, and we still want to keep track of the distribution of partial permutations. We will do that by studying partial representations of a semigroup $Q_n \supset \mathcal{F}_n$.

Let Q_n denote the semigroup generated by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, subject to the relations

$$b_i a_i b_i = b_i; \tag{2}$$

$$a_i b_i a_i = a_i; \tag{3}$$

$$b_i a_j = a_j b_i \qquad (i \neq j); \tag{4}$$

$$b_i b_j = b_j b_i. \tag{5}$$

Further, let Q'_n denote the semigroup obtained from Q_n by adding the relation

$$b_i a_i = 1 \tag{6}$$

(equivalently Q'_n is defined by (4), (5) and (6)). There is a natural projection $Q_n \to Q'_n$.

Lemma 2.2. *In the semigroup Q_n, for any distinct $i_1, \ldots, i_t$, we have that*

$$(a_{i_1} b_{i_1}) \cdots (a_{i_t} b_{i_t}) = a_{i_1} \cdots a_{i_t} b_{i_1} \cdots b_{i_t}.$$

Proof. By induction on t and the relation (4),

$$(a_{i_1} b_{i_1}) \cdots (a_{i_t} b_{i_t}) = a_{i_1} \cdots a_{i_{t-1}} b_{i_1} \cdots b_{i_{t-1}} a_{i_t} b_{i_t} = a_{i_1} \cdots a_{i_{t-1}} a_{i_t} b_{i_1} \cdots b_{i_{t-1}} b_{i_t}.$$

□

Proposition 2.3. *The map $\mathcal{F}_n \to Q_n$, sending $e_i \mapsto a_i b_i$, is a well-defined embedding.*

Proof. We first show that the map $\mathcal{F}_n \to Q_n$ is well defined. Since (1) gives a complete set of defining relations for $\mathcal{F}_n$, it suffices to verify its image in Q_n. For any distinct indices $i_1, \ldots, i_t$ we have by Lemma 2.2, (5) and (2) that:

$$\begin{aligned}
(a_{i_1} b_{i_1})(a_{i_2} b_{i_2}) \cdots (a_{i_t} b_{i_t})(a_{i_1} b_{i_1}) &= a_{i_1} \cdots a_{i_t} b_{i_1} \cdots b_{i_t} a_{i_1} b_{i_1} \\
&= a_{i_1} \cdots a_{i_t} b_{i_2} \cdots b_{i_t} b_{i_1} a_{i_1} b_{i_1} \\
&= a_{i_1} \cdots a_{i_t} b_{i_2} \cdots b_{i_t} b_{i_1} \\
&= a_{i_1} \cdots a_{i_t} b_{i_1} b_{i_2} \cdots b_{i_t} \\
&= (a_{i_1} b_{i_1})(a_{i_2} b_{i_2}) \cdots (a_{i_t} b_{i_t}).
\end{aligned}$$

Next, in Q'_n there are reduction rules for every instance of $b_i a_j$, so every element in Q'_n can be uniquely presented as a product $a_{j_1} \cdots a_{j_s} b_{j'_1} \cdots b_{j'_{s'}}$ where $j_1, \ldots, j_s$ are arbitrary and $j'_1 \leq \cdots \leq j'_{s'}$. It follows that ranging over distinct indices $i_1, \ldots, i_t$, the elements $(a_{i_1} b_{i_1})(a_{i_2} b_{i_2}) \cdots (a_{i_t} b_{i_t}) = a_{i_1} \cdots a_{i_t} b_{i_1} \cdots b_{i_t}$ are distinct in Q'_n, so the composition of maps $\mathcal{F}_n \to Q_n \to Q'_n$ is injective. □

Remark 2.4. *The proof above shows that Q'_n (and thus Q_n) contains a free semigroup $\langle a_1, \ldots, a_n \rangle$.*

2.3. **The sum of generators.** Let F be a field. After embedding $\mathcal{F}_n \hookrightarrow Q_n$ in Proposition 2.3, we are interested in the sum $s = e_1 + \cdots + e_n$, an element of the semigroup algebra $F[Q_n]$. The minimal polynomial of weighted sums $\sum \omega_i e_i$, in its action on the Tsetlin library of Example 2.1, can be computed from the representation theory of $\mathcal{F}_n$ [9, Section 15.5], but a proof over Q_n requires less machinery.

Proposition 2.5. *The element s satisfies the polynomial*

$$h_n(\lambda) = \lambda(\lambda - 1)\cdots(\lambda - n).$$

(The action of s on the Tsetlin library satisfies $h_n(\lambda)/(\lambda - (n-1))$; but this is irrelevant for us).

Proof. An element of Q_n is **k-standard** if it has the form $e_S = a_{i_1}\cdots a_{i_k} b_{i_1}\cdots b_{i_k}$ where $S = (i_1,\dots,i_k)$ is an ordered set with distinct indices. There are $\frac{n!}{(n-k)!}$ k-standard monomials, corresponding to ordered subsets of cardinality k.

We claim that $e_S e_{S'} = e_{S\cup S'}$. Indeed, for every common index $i \in S \cap S'$, one can "push" the first and second appearances of b_i until they are adjacent to the a_i in between, and then replace $b_i a_i b_i$ by b_i by (2) and move this generator to the right-hand side of the product.

Fix $\ell \leq k, k'$. There are $\binom{n}{\ell\,,\,k-\ell\,,\,k'-\ell\,,\,n-(k+k'-\ell)} k!k'!$ pairs of unordered sets (S, S') of cardinalities k, k' with $|S \cap S'| = \ell$, corresponding to choosing the index sets $S \cap S'$, $S - S'$ and $S' - S$, and then ordering S and S'.

For $k = 1,\dots,n$, let c_k denote the sum of all the k-standard monomials. In particular $s = c_1$. By the previous argument, $c_k c_{k'} = \sum_\ell \binom{k}{\ell}\binom{k'}{\ell}\ell! c_{k+k'-\ell}$, where the sum is over $\ell = \max\{0, k+k'-n\},\dots,\min\{k,k'\}$.

In particular, for $k = 1,\dots,n-1$ we have that $c_1 c_k = c_{k+1} + k c_k$; and $c_1 c_n = n c_n$. It follows that $(c_1 - k)c_k = c_{k+1}$, so $0 = (c_1 - n)c_n = (c_1 - n)(c_1 - (n-1))c_{n-1} = \cdots = (c_1 - n)\cdots(c_1 - 1)c_1$, proving our claim. □

3. Partial representations of Q_n

We now discuss partial representations of Q_n, the semigroup introduced in the previous section, in connection with (full) representations of $\mathcal{F}_n$.

3.1. **Partial transformations.** Let V be a vector space. A **partial linear transformation** $f : V \to V$ is a linear transformation defined on a subspace $\mathrm{dom}(f)$. The composition $f \circ g$ of partial linear transformations has domain $g^{-1}(\mathrm{dom}(f))$, so it is defined by $(f \circ g)(x) = f(g(x))$ where both evaluations in the right-hand side make sense. The set $\mathrm{PEnd}(V)$ of partial linear transformations is thus a monoid under composition, where the identity map serves as the identity element. The subgroup of invertible elements is $\mathrm{GL}(V)$. Every partial zero map is a right zero in the monoid, so there are no left zeros (unless $V = 0$). See, for example, [8]. Partial transformations (regardless of linearity) are discussed in [9, Section 5.6].

$$\begin{tikzcd} & P_S \arrow[dr, "\pi_i"] & \\ P_{S-\{j\}} \arrow[ur, "\mu_j"] \arrow[dr, "\pi_i"'] & & P_{S-\{i\}} \\ & P_{S-\{i,j\}} \arrow[ur, "\mu_j"'] & \end{tikzcd} \qquad \begin{tikzcd} & P_S \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_i"] & \\ P_{S-\{j\}} \arrow[dr, "\pi_i"'] & & P_{S-\{i\}} \arrow[dl, "\pi_j"] \\ & P_{S-\{i,j\}} & \end{tikzcd}$$

FIGURE 1. Conditions (3b) and (3c) for inclusion-exclusion systems

A **partial representation** of a semigroup G over the field F is a homomorphism from G to $\mathrm{PEnd}(V)$, where V is a vector space over F. A **morphism** from $\rho : G \to \mathrm{PEnd}(V)$ to $\rho' : G \to \mathrm{PEnd}(V')$ is a linear transformation $\psi : V \to V'$ such that for every $g \in G$ and every $x \in \mathrm{dom}(\rho g)$, $\psi((\rho g)x) = (\rho' g)(\psi x)$. This defines a category $\mathbf{PRep}(G)$ of partial representations of G over F. The classical category $\mathbf{Rep}(G)$ of (full) representations is a subcategory of $\mathbf{PRep}(G)$. For example, if G is a group then $\mathbf{PRep}(G) = \mathbf{Rep}(G)$.

3.2. **Inclusion-exclusion systems.** Any partial representation of Q_n induces a partial representation of the subsemigroup $\mathcal{F}_n$. We construct partial representations with "built-in" homogeneity, which restrict in a natural way to a *full* representation of $\mathcal{F}_n$. We will use the term **index set** for subsets of $\{1, \dots, n\}$.

Definition 3.1. *An* **inclusion-exclusion system** $(\{P_S\}, \{\pi_j, \mu_j\})$ *(of order n) is composed of vector spaces P_S for each subset $S \subseteq \{1, \dots, n\}$, and partial linear transformations $\pi_1, \dots, \pi_n$ and $\mu_1, \dots, \mu_n$ on the direct sum $\mathbf{P} = \bigoplus P_S$, such that:*

(1) *For every j:*
  (a) *the domain of π_j is $\bigoplus_{j \in S} P_S$, and*
  (b) *the domain of μ_j is $\bigoplus_{j \in S} P_{S-\{j\}}$;*
(2) *For every j and every subset S containing j:*
  (a) *π_j restricts to a map $P_S \to P_{S-\{j\}}$, and*
  (b) *μ_j restricts to a map $P_{S-\{j\}} \to P_S$;*
(3) *The maps satisfy:*
  (a) *$\pi_j \mu_j$ is the identity on each summand $P_{S-\{j\}}$ ($j \in S$),*
  (b) *$\pi_i \mu_j = \mu_j \pi_i$ ($i \neq j$) and*
  (c) *$\pi_i \pi_j = \pi_j \pi_i$.*

Notice that (3a) does not mean that $\pi_j \mu_j = 1$; indeed, $\pi_j \mu_j$ is a partial identity, defined on the components $P_{S-\{j\}}$. This equality does imply that $\pi_j \mu_j \pi_j = \pi_j$ and $\mu_j \pi_j \mu_j = \mu_j$ as partial linear transformations on the full space $\mathbf{P}$. Likewise, (3b) and (3c) are equalities of partial linear transformations (defined on the same subspace, and equal when defined). Figure 1 depicts conditions (3b) and (3c).

It is useful to view the maps π_j as going "downwards" from **the top space** $P_{\{1,\dots,n\}}$ of the system, and the μ_j as going "upwords". The relations provide the following:

Proposition 3.2. *An inclusion-exclusion system defines a partial representation of Q_n, by sending $a_j \mapsto \mu_j$ and $b_j \mapsto \pi_j$.*

Perhaps more significantly:

Proposition 3.3. *An inclusion-exclusion system induces a full representation of $\mathcal{F}_n$ on the space $P_{\{1,\dots,n\}}$ by letting the generators e_j act via $\epsilon_j = \mu_j\pi_j$.*

Proof. Since $e_j \mapsto a_jb_j$ defines an embedding $\mathcal{F}_n \hookrightarrow Q_n$ by Proposition 2.3, we clearly have a partial representation of $\mathcal{F}_n$ on $\mathbf{P} = \bigoplus P_S$. But the generators ϵ_j are fully defined on the top space, so the restriction to the top space is a full representation. □

We say that an inclusion-exclusion system $\{P_S\}$ of order n **extends** a representation $\rho\colon\mathcal{F}_n \to \mathrm{End}(V)$ if $V = P_{\{1,\dots,n\}}$ is the top space, and the induced representation of Proposition 3.3 coincides with ρ. We offer a natural extension of the Tsetlin representation from Example 2.1.

Example 3.4. *Consider the "Tsetlin inclusion-exclusion system". Fix a set Y (which could be empty). For any subset $S = \{i_1,\dots,i_k\}$ of $\{1,\dots,n\}$, let P_S be the space of multilinear polynomials in the letters $\{x_{i_1},\dots,x_{i_k}\} \cup Y$.*

The map π_j is defined on polynomials containing x_j, by substituting $x_j \mapsto 1$. The map μ_j is defined on polynomials not containing x_j, by $\mu_j(f) = fx_j$.

This example continues Example 2.1: π_j is now the operation of taking out the book x_j (assuming it is on the shelf), and μ_j is the operation of returning x_j to the back of the shelf (when it is not already on the shelf). It is easy to verify that:

(a) $\pi_j\mu_j$ is the identity on each summand $P_{S-\{j\}}$ ($j \in S$);
(b) the operations of taking out x_i and returning x_j ($j \neq i$) commute; and
(c) when taking out two books, the order of the removals does not matter.

We thus have verified that Example 3.4 is indeed an inclusion-exclusion system. Notice that $\mu_i\mu_j \neq \mu_j\mu_i$, because the order of the books at the end of the shelf is x_jx_i in one case, and x_ix_j in the other.

4. Extending representations

In this section we define the category $\mathbf{IE}(Q_n)$ of inclusion-exclusion systems, and show that it is equivalent to the category $\mathbf{Rep}(\mathcal{F}_n)$ of representations of $\mathcal{F}_n$.

4.1. **Extensions.** We first show that every representation of $\mathcal{F}_n$ extends to an inclusion-exclusion system.

Proposition 4.1. *Let $\epsilon_1,\dots,\epsilon_t \in \mathrm{End}(V)$ be idempotents generating a left regular band. The kernel $\mathrm{Ker}(\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)})$ is independent of the permutation $\sigma \in S_t$.*

Proof. If $\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)}x = 0$ then $\epsilon_{\sigma(t)}\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t-1)}x = \epsilon_{\sigma(t)}\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)}x = 0$ by the identity $uvu = uv$, which proves that $\mathrm{Ker}(\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)})$ is invariant under cyclic permutations of the indices. By the same argument applied to the first two idempotents, if $\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)}x = 0$ we also have that $\epsilon_{\sigma(2)}\epsilon_{\sigma(1)}\epsilon_{\sigma(3)}\cdots\epsilon_{\sigma(t)}x = 0$, so that the kernel is invariant under transpositions of the first two indices. But together these two operations generate the symmetric group S_t. $\square$

Needless to say, the product $\epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)}$ itself does depend on σ.

Corollary 4.2. *Let $\epsilon_1, \dots, \epsilon_t \in \mathrm{End}(V)$ be idempotents generating a left regular band. For every permutation $\sigma \in S_t$, the map $\epsilon_1\cdots\epsilon_t x \mapsto \epsilon_{\sigma(1)}\cdots\epsilon_{\sigma(t)}x$ is well-defined on $\epsilon_1\cdots\epsilon_t V$.*

We will let $\bar{S}$ denote the complement of an index set S with respect to $\{1, \dots, n\}$.

Theorem 4.3. *Every representation of $\mathcal{F}_n$ extends to an inclusion-exclusion system.*

Proof. We are given a representation of $\mathcal{F}_n$ on a vector space V, generated by idempotent maps $\epsilon_1, \dots, \epsilon_n$ on V. For any index set $S = \{i_1, \dots, i_t\}$, where $i_1 < \cdots < i_t$, write $\epsilon_S = \epsilon_{i_1}\cdots\epsilon_{i_t}$.

We define an inclusion-exclusion system as follows. For every subset S, we take

$$P_S = \epsilon_{\bar{S}}V. \tag{7}$$

Thus the top space is $P_{\{1,\dots,n\}} = \epsilon_\emptyset V = V$, and the bottom space is $P_\emptyset = \epsilon_1\cdots\epsilon_n V$. Next, let j be an index. For an index set S containing j, we define the operator $\pi_j : P_S \to P_{S-\{j\}}$ by

$$\pi_j(\epsilon_{\bar{S}}x) = \epsilon_{\bar{S}\cup\{j\}}x;$$

inserting, so to speak, ϵ_j in its natural place among the idempotents indexed by S. This map is well-defined because $\epsilon_j\epsilon_{\bar{S}}x$ is clearly determined by $\epsilon_{\bar{S}}x$, and we can then apply Corollary 4.2 to reorder and obtain $\epsilon_{\bar{S}\cup\{j\}}x$. Similarly, we define $\mu_j : P_{S-\{j\}} \to P_S$ by

$$\mu_j(\epsilon_{\bar{S}\cup\{j\}}x) = \epsilon_{\bar{S}}\epsilon_j x,$$

which is again well-defined by Corollary 4.2. Here we simply push ϵ_j to the right, retaining the other operators in their original order.

It remains to verify the relations (3a)–(3c) of Definition 3.1. Let S be an index set containing j. For every $x \in V$ we have that

$$\pi_j\mu_j(\epsilon_{\bar{S}\cup\{j\}}x) = \pi_j\epsilon_{\bar{S}}(\epsilon_j x) = \epsilon_{\bar{S}\cup\{j\}}\epsilon_j x = \epsilon_{\bar{S}\cup\{j\}}x$$

by the identity $uvu = uv$, which proves (3a). For (3b) we need to verify that $\pi_i\mu_j = \mu_j\pi_i$ on $P_{S-\{j\}}$, where i, j are distinct indices. Indeed, by definition

$$\pi_i\mu_j\epsilon_{\bar{S}\cup\{j\}}x = \pi_i\epsilon_{\bar{S}}\epsilon_j x = \epsilon_{\bar{S}\cup\{i\}}\epsilon_j x = \mu_j\epsilon_{\bar{S}\cup\{i,j\}}x = \mu_j\pi_i\epsilon_{\bar{S}\cup\{j\}}x.$$

Moreover if $i, j \in S$ then

$$\pi_i\pi_j\epsilon_{\bar{S}}x = \pi_i\epsilon_{\bar{S}\cup\{j\}}x = \epsilon_{\bar{S}\cup\{i,j\}}x = \pi_j\epsilon_{\bar{S}\cup\{i\}}x = \pi_j\pi_i\epsilon_{\bar{S}}x,$$

so (3c) holds and we constructed an inclusion-exclusion system.

Finally, for every $j \in S$ we have that

$$\mu_j \pi_j \epsilon_{\bar{S}} x = \mu_j \epsilon_{\bar{S} \cup \{j\}} x = \epsilon_{\bar{S}} \epsilon_j x;$$

in particular for $S = \{1, \dots, n\}$ we obtain that $\mu_j \pi_j = \epsilon_j$ on the top level, so the system extends the given representation, as asserted. $\square$

4.2. **Morphisms.** Let $\mathbf{P} = (\{P_S\}, \{\pi_j, \mu_j\})$ and $\mathbf{P}' = (\{P'_S\}, \{\pi'_j, \mu'_j\})$ be inclusion-exclusion systems of the same order n. A **morphism** $\psi : \mathbf{P} \to \mathbf{P}'$ is a set of linear transformations $\psi_S : P_S \to P'_S$ such that for every index set S and every $j \in S$, the following diagrams commute:

$$\begin{tikzcd}
P_S \arrow[r, "\psi_S"] \arrow[d, "\pi_j"] & P'_S \arrow[d, "\pi'_j"] & & P_S \arrow[r, "\psi_S"] & P'_S \\
P_{S-\{j\}} \arrow[r, "\psi_{S-\{j\}}"] & P'_{S-\{j\}} & & P_{S-\{j\}} \arrow[r, "\psi_{S-\{j\}}"] \arrow[u, "\mu_j"] & P'_{S-\{j\}} \arrow[u, "\mu'_j"]
\end{tikzcd}$$

Composition of morphisms is defined in the obvious manner. We thus have defined the category $\mathbf{IE}(Q_n)$, whose objects are inclusion-exclusion systems of order n. Since every inclusion-exclusion system is in particular a partial representation of Q_n, $\mathbf{IE}(Q_n)$ is a sub-category of $\mathbf{PRep}(Q_n)$ defined above.

One can restrict an inclusion-exclusion system to the representation of $\mathcal{F}_n$ on the top space, and a morphism ψ of inclusion-exclusion systems to a morphism $\psi_{1,\dots,n}$ of the top spaces as representations of $\mathcal{F}_n$. We say that ψ is an **extension** of $\psi_{1,\dots,n}$. It follows from the next remark that restriction defines a functor $\mathbf{IE}(Q_n) \to \mathbf{Rep}(\mathcal{F}_n)$:

Remark 4.4. *Let $\phi : V \to V'$ and $\phi' : V' \to V''$ be morphisms of $\mathcal{F}_n$-representations. If ψ, ψ' are extensions of ϕ, ϕ', then $\psi \circ \psi'$ is an extension of $\phi \circ \phi'$.*

The following theorem is the key in relating the two categories.

Theorem 4.5. *Let $\phi : V \to V'$ be a morphism of representations of $\mathcal{F}_n$. Let $\mathbf{P}$ and $\mathbf{P}'$ be inclusion-exclusion systems extending V and V' respectively. Then there is a unique morphism $\psi : \mathbf{P} \to \mathbf{P}'$ extending ϕ.*

Proof. Write $\mathbf{P} = (\{P_S\}, \{\pi_j, \mu_j\})$ and $\mathbf{P}' = (\{P'_S\}, \{\pi'_j, \mu'_j\})$. Let $\epsilon_1, \dots, \epsilon_n$ and $\epsilon'_1, \dots, \epsilon'_n$ be the generating idempotents for the given representations of $\mathcal{F}_n$ on V and V', respectively. Thus $\epsilon_j = \mu_j \pi_j$ and $\epsilon'_j = \mu'_j \pi'_j$ on the top components $V = P_{\{1,\dots,n\}}$ and $V' = P'_{\{1,\dots,n\}}$.

For any index set S, write

$$\pi_S = \pi_{i_1} \cdots \pi_{i_t} \tag{8}$$

and

$$\mu_S = \mu_{i_1} \cdots \mu_{i_t} \tag{9}$$

where $S = \{i_1, \dots, i_t\}$ is the set in increasing order; and similarly for π'_S and μ'_S. By Lemma 2.2 we have that $\epsilon_S = \mu_S \pi_S$, and $\epsilon'_S = \mu'_S \pi'_S$. Since the π_j are surjective on each component where they are defined, we have that $P_{\bar{S}} = \pi_S V$. And since the μ_j are injective, we have that $\mathrm{Ker}(\pi_S) = \mathrm{Ker}(\epsilon_S)$, which is contained in $\mathrm{Ker}(e'_S) = \mathrm{Ker}(\pi'_S)$ because ϕ is a morphism of representations. We can therefore define $\psi_{\bar{S}} : P_{\bar{S}} \to P'_{\bar{S}}$ by

$$\psi_{\bar{S}}(\pi_S x) = \pi'_S \phi(x).$$

These maps clearly satisfy commutativity with respect to the π_j and π'_j, and are inductively defined by this condition. It remains to prove commutativity with respect to μ_j and μ'_j. Let S be an index set containing j. Let $x \in V$, so that $\pi_{\bar{S} \cup \{j\}} x \in P_{S - \{j\}}$. We need to prove that $\psi_S \mu_j (\pi_{\bar{S} \cup \{j\}} x) = \mu'_j \psi_{S-\{j\}} (\pi_{\bar{S} \cup \{j\}} x)$. By definition of ψ_S we have that

$$\psi_S \mu_j (\pi_{\bar{S} \cup \{j\}} x) = \psi_S \pi_{\bar{S}} \mu_j \pi_j x = \pi'_{\bar{S}} \phi \, \mu_j \pi_j x = \pi'_{\bar{S}} \phi \, \epsilon_j x,$$

where we have used $\mu_j \pi_i = \pi_i \mu_j$ for $i \neq j$. On the other hand

$$\mu'_j \psi_{S-\{j\}} (\pi_{\bar{S} \cup \{j\}} x) = \mu'_j \pi'_{\bar{S} \cup \{j\}} \phi \, x = \pi'_{\bar{S}} \mu'_j \pi'_j \phi \, x = \pi'_{\bar{S}} \epsilon'_j \phi \, x,$$

and these are equal because $\phi(\epsilon_j x) = \epsilon'_j \phi(x)$. □

Theorem 4.6. *Every representation of $\mathcal{F}_n$ extends to a unique inclusion-exclusion system.*

Proof. Existence is Theorem 4.3. Let V be a representation of $\mathcal{F}_n$, and let $\mathbf{P}$ and $\mathbf{P'}$ be extensions. By Theorem 4.5 there are unique extensions $\psi : \mathbf{P} \to \mathbf{P'}$ and $\psi' : \mathbf{P'} \to \mathbf{P}$ of the identity map $V \to V$, so by Remark 4.4, $\psi' \circ \psi$ is the identity morphism of $\mathbf{P}$ and $\psi \circ \psi'$ the identity morphism of $\mathbf{P'}$, implying that ψ is an isomorphism. □

In particular, the "dimension cube" of an inclusion-exclusion system $\{P_S\}$, namely the function $S \mapsto \dim(P_S)$, is determined by the representation of $\mathcal{F}_n$ on the top component:

Proposition 4.7. *Let V be a representation space of $\mathcal{F}_n$, generated by endomorphisms $\epsilon_1, \dots, \epsilon_n$. Let $(\{P_S\}, \{\pi_j, \mu_j\})$ be an extension of V. Then, for any index set $S = \{i_1, \dots, i_t\}$, $\dim P_{\bar{S}} = \dim((\epsilon_{i_1} \cdots \epsilon_{i_t})V)$.*

Proof. In the extension constructed in Theorem 4.3, the dimensions are as stated by (7). This extension is unique up to isomorphism by Theorem 4.6, so these are the dimensions in any extension. □

5. The complex

We are interested in the **absolute kernel** of a representation V of the semigroup $\mathcal{F}_n$, which is the intersection of the kernels of all the elements $1 \neq e \in \mathcal{F}_n$, equivalently the intersection $K = \bigcap \mathrm{Ker}(\epsilon_j)$ where $\epsilon_1, \dots, \epsilon_n$ are the maps defined by the generators $e_1, \ldots, e_n$ of $\mathcal{F}_n$. We study this space by constructing a resolution of K, starting from $0 \to K \to V$.

Let $(\{P_S\}, \{\pi_j, \mu_j\})$ be the inclusion-exclusion system extending V (which exists by Theorem 4.6). In particular $\mu_j \pi_j = \epsilon_j$. Since each μ_j is injective, we also have that

$$K = \bigcap \mathrm{Ker}(\pi_j), \tag{10}$$

where the maps are considered when they are all defined, namely on the top space $V = P_{\{1,\ldots,n\}}$.

We fix the following notation:

Notation 5.1. *For $i_1 < \cdots < i_k$, always in the range $[1, n]$, we write $P_{i_1,\ldots,i_k}$ for the space $P_{\{i_1,\ldots,i_k\}}$. For $k = 0, \ldots, n$, let $\mathbf{P}_k = \bigoplus_{i_1<\cdots<i_k} P_{i_1,\ldots,i_k}$ be the direct sum, composed of elements $f = \sum f_{i_1,\ldots,i_k}$ for vectors $f_{i_1,\ldots,i_k} \in P_{i_1,\ldots,i_k}$. In particular $\mathbf{P}_0 = F$ and $\mathbf{P}_n = P_{1,\ldots,n}$.*

We tautologically set each component $f_{i_1,\ldots,i_k}$ of an element $f \in \mathbf{P}_k$ to be *an alternating function* of the indices. This agreement is critical to the symbol manipulation throughout the paper.

The partial maps π_j and μ_j respect the grading $\mathbf{P} = \bigoplus_{k=0}^{n} \mathbf{P}_k$, in the sense that for every $k > 0$, π_j is a partial linear transformation $\mathbf{P}_k \to \mathbf{P}_{k-1}$, and μ_j a partial linear transformation $\mathbf{P}_{k-1} \to \mathbf{P}_k$. In what follows we are always careful to apply those maps only where they are defined.

Because of the equality (10), the absolute kernel K is equal to the kernel of the map $\partial_n : \mathbf{P}_n \to \mathbf{P}_{n-1}$ defined by $\partial_n(f) = \sum_{j=0}^{n-1} (-1)^j \pi_j(f)$. This observation leads us to define the chain complex

$$0 \longrightarrow K \hookrightarrow \mathbf{P}_n \xrightarrow{\partial_n} \mathbf{P}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} \mathbf{P}_1 \xrightarrow{\partial_1} \mathbf{P}_0 \longrightarrow 0 \tag{11}$$

where the boundary maps $\partial_{k+1} : \mathbf{P}_{k+1} \to \mathbf{P}_k$ are defined for $k = 0, \ldots, n-1$ by

$$\partial_{k+1}\Big(\sum_{i_0<\cdots<i_k} f_{i_0,\ldots,i_k}\Big) = \sum_{i_0<\cdots<i_k} \sum_{j=0}^{k} (-1)^j \pi_{i_j}(f_{i_0,\ldots,i_k}). \tag{12}$$

Proposition 5.2. *(11) is indeed a chain complex.*

Proof. A standard argument, using the fact that the π_j commute, shows that

$$\partial_k \circ \partial_{k+1} = 0. \tag{13}$$

□

Our goal in the coming sections is:

Theorem 5.3. *The complex (11) is exact, provided that $n!$ is nonzero in F.*

Remark 5.4. *In general (11) is a complex of vector spaces. Nevertheless, the symmetric group S_n acts on $\mathcal{F}_n$ and on Q_n by permuting the generators, and for our key example of the Tsetlin library, Example 3.4, the representation is S_n-equivariant.*

Since the boundary maps (and all other maps defined below) are symmetric with respect to this external operation on the spaces $\mathbf{P}_k$ and the operators π_j, we obtain in this case a complex of S_n-modules.

Proposition 5.5. *Writing $f = \sum_{i_0<\cdots<i_k} f_{i_0,\dots,i_k}$ (for some $k = 0,\dots,n-1$), we have that*

$$(\partial_{k+1}f)_{r_1,\dots,r_k} = \sum_{t\notin\{r_1,\dots,r_k\}} \pi_t(f_{t,r_1,\dots,r_k}).$$

Proof. We are given an element $f \in \mathbf{P}_{k+1}$. Fix $r_1 < \cdots < r_k$. A summand $(-1)^j\pi_{i_j}(f_{i_0,\dots,i_k})$ in the right-hand side of (12) falls into the component $P_{r_1,\dots,r_k}$ precisely when $(i_0,\dots,\hat{i_j},\dots,i_k) = (r_1,\dots,r_k)$. This is the case when, for some $j = 0,\dots,k$ and $t \in (r_j, r_{j+1})$ (formally setting $r_0 = 0$ and $r_{k+1} = n+1$), we have that $(i_0,\dots,i_{j-1},i_j,i_{j+1},\dots,i_k) = (r_1,\dots,r_j,t,r_{j+1},\dots,r_k)$; and then the entry is

$$(-1)^j\pi_{i_j}(f_{i_0,\dots,i_k}) = (-1)^j\pi_t(f_{r_1,\dots,r_j,t,r_{j+1},\dots,r_k}) = \pi_t(f_{t,r_1,\dots,r_j,r_{j+1},\dots,r_k}).$$

□

6. Local Tsetlin libraries

Fix $k = 0,\dots,n$, and define the map $s_k : \mathbf{P}_k \to \mathbf{P}_k$ by

$$(s_k f)_{p_1,\dots,p_k} = \sum_i \mu_{p_i}\pi_{p_i}(f_{p_1,\dots,p_k}). \tag{14}$$

This map acts on each summand $P_{p_1,\dots,p_k}$, because the $(p_1,\dots,p_k)$th entry of $s_k f$ is defined in terms of the same entry of f. The idempotents $\epsilon_{p_i} = \mu_{p_i}\pi_{p_i}$ define a representation of $\mathcal{F}_k$ on the space $P_{p_1,\dots,p_k}$. By Proposition 2.5, we have that s_k satisfies the polynomial

$$h_k(\lambda) = \lambda(\lambda-1)\cdots(\lambda-k). \tag{15}$$

The s_k participate in the following connection with the differntials:

Proposition 6.1. *We have (for $k = 1,\dots,n$) that*

$$\partial_k \circ s_k - s_{k-1} \circ \partial_k = \partial_k. \tag{16}$$

Proof. Let $f \in \mathbf{P}_k$, and compute by Proposition 5.5:

$$\begin{aligned}
\partial_k(s_k f) &= \sum_{r_1<\cdots<r_{k-1}} \sum_{t\notin\{r_1,\dots,r_{k-1}\}} \pi_t((s_k f)_{t,r_1,\dots,r_{k-1}}) \\
&= \sum_{r_1<\cdots<r_{k-1}} \sum_{t\notin\{r_1,\dots,r_{k-1}\}} \pi_t\left(\mu_t\pi_t(f_{t,r_1,\dots,r_{k-1}}) + \sum_{i=1}^{k-1}\mu_{r_i}\pi_{r_i}(f_{t,r_1,\dots,r_{k-1}})\right) \\
&= \sum_{r_1<\cdots<r_{k-1}} \sum_{t\notin\{r_1,\dots,r_{k-1}\}} \left(\pi_t(f_{t,r_1,\dots,r_{k-1}}) + \sum_{i=1}^{k-1}\mu_{r_i}\pi_t\pi_{r_i}(f_{t,r_1,\dots,r_{k-1}})\right) \\
&= \sum_{r_1<\cdots<r_{k-1}} \Bigg[\sum_{t\notin\{r_1,\dots,r_{k-1}\}} \pi_t(f_{t,r_1,\dots,r_{k-1}}) \\
&\qquad + \sum_{i=1}^{k-1}\mu_{r_i}\pi_{r_i}\left(\sum_{t\notin\{r_1,\dots,r_{k-1}\}} \pi_t(f_{t,r_1,\dots,r_{k-1}})\right)\Bigg] \\
&= \sum_{r_1<\cdots<r_{k-1}} \left[(\partial_k f)_{r_1,\dots,r_{k-1}} + \sum_{i=1}^{k-1}\mu_{r_i}\pi_{r_i}\left((\partial_k f)_{r_1,\dots,r_{k-1}}\right)\right] \\
&= (\partial_k f) + s_{k-1}(\partial_k f).
\end{aligned}$$

□

Let $Z_k = \operatorname{Ker}(\partial_k) = \{f \in \mathbf{P}_k \mid \partial_k f = 0\}$ and $B_k = \operatorname{Im}(\partial_{k-1})$, so as usual, by (13), we have that $B_k \subseteq Z_k$.

Proposition 6.2. *The map $s_k : \mathbf{P}_k \to \mathbf{P}_k$ preserves both B_k and Z_k.*

Proof. Suppose $\partial_k f = 0$. Then $0 = (\partial_k \circ s_k - s_{k-1}\circ\partial_k - \partial_k)f = \partial_k(s_k f)$. Next suppose $f = \partial_{k+1}g$ for some $g \in \mathbf{P}_k$. Then $s_k f = s_k\partial_{k+1}g = \partial_{k+1}(s_{k+1}g - g) \in \operatorname{Im}(\partial_{k+1})$. □

7. Going up

For $k = 0,\dots,n-1$, define a map $\theta_k : \mathbf{P}_k \to \mathbf{P}_{k+1}$ by

$$\theta_k f = \sum_{i_0<\cdots<i_k} \left[\sum_{j=0}^{k}(-1)^j\mu_{i_j}(f_{i_0,\dots,\hat{i_j},\dots,i_k})\right]. \tag{17}$$

When a set $r_* = \{r_1,\dots,r_{k-1}\}$ is fixed, we let $j(x)$ be defined as the only index for which $r_{j(x)} < x < r_{j(x)+1}$, with $j(x) = 0$ for $x < r_1$ and $j(x) = k-1$ for $r_{k-1} < x$.

Proposition 7.1. *Let $k = 1,\dots,n$. For $g \in \mathbf{P}_{k-1}$,*

$$\theta_{k-1}g = \sum_{r_1<\cdots<r_{k-1}} \sum_{s\notin r_*} (-1)^{j(s)}\mu_s(g_{r_1,\dots,r_{k-1}}).$$

Proof. Fix $i_1 < \cdots < i_k$. A summand $(-1)^{j(s)}\mu_s(f_{r_1,\dots,r_{k-1}})$ belongs to the component $P_{i_1,\dots,i_k}$ if and only if $(r_1,\dots,r_{j(s)},s,r_{j(s)+1},\dots,r_k) = (i_1,\dots,i_k)$, in which case it is equal to $(-1)^{j-1}\mu_{i_j}(g_{i_1,\dots,\hat{i_j},\dots,i_k})$ for $j = j(s)+1$. But indeed $\theta_{k-1}g = \sum_{i_1<\dots<i_k}\left[\sum_{j=1}^k(-1)^{j-1}\mu_{i_j}(g_{i_1,\dots,\hat{i_j},\dots,i_k})\right]$, where $(-1)^{j-1}$ replaces $(-1)^j$ in (17) because we are ranging over $i_1<\dots<i_k$ rather than $i_0<\dots<i_{k-1}$. □

Proposition 7.2. *Let $k = 1,\dots,n-1$. Then*

$$\partial_{k+1}\theta_k + \theta_{k-1}\partial_k = s_k + (n-k).$$

Proof. In the computation below, when a set $r_* = \{r_1,\dots,r_{k-1}\}$ is fixed, we understand the sum over s and t as ranging over the indices $1,\dots,n$, avoiding this set. Let $f \in \mathbf{P}_k$. By definition,

$$\begin{aligned}
\partial_{k+1}\theta_k f &= \sum_{i_0<\dots<i_k}\sum_{j=0}^k(-1)^j\pi_{i_j}\left[\sum_{j'=0}^k(-1)^{j'}\mu_{i_{j'}}(f_{i_0,\dots,\hat{i_{j'}},\dots,i_k})\right]\\
&= \sum_{i_0<\dots<i_k}\left(\sum_{j\neq j'}(-1)^{j+j'}\pi_{i_j}\mu_{i_{j'}}\left(f_{i_0,\dots,\hat{i_{j'}},\dots,i_k}\right) + \sum_{j=0}^k\pi_{i_j}\mu_{i_j}\left(f_{i_0,\dots,\hat{i_j},\dots,i_k}\right)\right)\\
&= \sum_{i_0<\dots<i_k}\left(\sum_{j\neq j'}(-1)^{j+j'}\mu_{i_{j'}}\pi_{i_j}(f_{i_0,\dots,\hat{i_{j'}},\dots,i_k})\right) + \sum_{i_0<\dots<i_k}\sum_{j=0}^k f_{i_0,\dots,\hat{i_j},\dots,i_k};
\end{aligned}$$

in the left-most sum we range over $k+1$ indices and select the indices of $\mu_{i_{j'}}$ and π_{i_j} from them; the same element is obtained by ranging over $k-1$ indices and then choosing the indices of μ_s and π_t from the complement. We thus have:

$$\begin{aligned}
&= \sum_{r_1<\dots<r_{k-1}}\sum_{s\neq t}\left((-1)^{j(s)+j(t)+1}\mu_s\pi_t(f_{r_1,\dots,r_{j(t)},t,r_{j(t)+1},\dots,r_{k-1}})\right) + (m-k)\sum_{r_1<\dots<r_k} f_{r_1,\dots,r_k}\\
&= \sum_{r_1<\dots<r_{k-1}}\sum_{s\neq t}\left((-1)^{j(s)+1}\mu_s\pi_t(f_{t,r_1,\dots,r_{k-1}})\right) + (m-k)\sum_{r_1<\dots<r_k} f_{r_1,\dots,r_k}\\
&= \sum_{r_1<\dots<r_{k-1}}\sum_{s\notin r_*}(-1)^{j(s)+1}\mu_s\left(\sum_{t\notin\{s\}\cup r_*}\pi_t(f_{t,r_1,\dots,r_{k-1}})\right) + (m-k)f\\
&= \sum_{r_1<\dots<r_{k-1}}\sum_{s\notin r_*}(-1)^{j(s)+1}\mu_s\left((\partial_k f)_{r_1,\dots,r_{k-1}} - \pi_s(f_{s,r_1,\dots,r_{k-1}})\right) + (m-k)f\\
&= \sum_{r_1<\dots<r_{k-1}}\sum_{s\notin r_*}(-1)^{j(s)+1}\mu_s(\partial_k f)_{r_1,\dots,r_{k-1}}\\
&\quad + \sum_{r_1<\dots<r_{k-1}}\sum_{s\notin r_*}(-1)^{j(s)}\mu_s\pi_s(f_{s,r_1,\dots,r_{k-1}}) + (m-k)f\\
&= -\theta_{k-1}\partial_k f + \sum_{p_1<\dots<p_k}\left[\sum_i \mu_{p_i}\pi_{p_i}(f_{p_1,\dots,p_k})\right] + (m-k)f\\
&= -\theta_{k-1}\partial_k f + s_k f + (n-k)f,
\end{aligned}$$

where in the second to last equality we applied Proposition 7.1 for $g = \partial_k f$. □

Corollary 7.3. *Let $k = 1,\dots,n-1$. If $f \in \mathbf{P}_k$ satisfies $\partial_k f = 0$, then*

$$\partial_{k+1}\theta_k f = (s_k + n - k)f.$$

8. Conclusion of the proof of exactness

Recall the polynomial $h_k(\lambda) = \prod_{i=0}^{k}(\lambda - i)$ from (15).

Lemma 8.1. *Fix $k = 1, \dots, n-1$. Then $\frac{h_k(\lambda)+(-1)^k h_k(n)}{\lambda+n-k}$ is a polynomial with integral coefficients.*

Proof. We claim that $\lambda+n-k$ divides $h_k(\lambda)+(-1)^k h_k(n)$ as a polynomial over the integers. Indeed, $k-n$ is a root of $h_k(\lambda)+(-1)^k h_k(n)$, as $h_k(k-n)+(-1)^k h_k(n) = \prod_{i=0}^{k}(k-n-i) + (-1)^k \prod_{i=0}^{k}(n-i) = \prod_{i=0}^{k}(k-n-i) - \prod_{i=0}^{k}(i-n) = 0$; and we are done by Gauss' lemma. $\square$

Notice that $h_k(n) = \prod_{i=0}^{k}(n-i) = \frac{n!}{(n-k-1)!}$. We can now wrap up the computations.

Proof of Theorem 5.3. Fix $k = 1, \dots, n-1$. By the lemma, $\beta(\lambda) = \frac{h_k(\lambda)+(-1)^k h_k(n)}{\lambda+n-k}$ has integral coefficients, and we write $(\lambda+n-k)\beta(\lambda) = h_k(\lambda) + (-1)^k \frac{n!}{(n-k-1)!}$. Projecting to F and taking the operator s_k (defined in Section 6) for λ, we have by Proposition 2.5 that $(s_k+n-k)\beta(s_k) = h_k(s_k) + (-1)^k \frac{n!}{(n-k-1)!} = (-1)^k \frac{n!}{(n-k-1)!}$.

Let $f \in \mathbf{P}_k$ be an element for which $\partial_k f = 0$. By Proposition 6.2, $\partial_k \beta(s_k) f = 0$ as well. Applying Corollary 7.3 we get

$$\frac{n!}{(n-k-1)!} f = (-1)^k (s_k+n-k)\beta(s_k) f = (-1)^k \partial_{k+1}\theta_k \beta(s_k) f \in \mathrm{Im}(\partial_{k+1}).$$

and since $n!$ is invertible by assumption, we have proved that $f \in \mathrm{Im}(\partial_{k+1})$. $\square$

In other words, (11) is a resolution of $\bigcap \mathrm{Ker}(\epsilon_j)$.

9. The case $n = 2$

In this section we specialize Theorem 5.3, on the exactness of the complex (11), to the case $n = 2$. Let $(\{P_{12}, P_1, P_2, P_\emptyset\}, \{\pi_1, \pi_2, \mu_1, \mu_2\})$ be an inclusion-exclusion system for Q_2, as depicted in Figure 2.

$$\begin{tikzcd}
& P_{12} \arrow[dl, shift right, "\pi_2"'] \arrow[dr, shift left, "\pi_1"] & \\
P_1 \arrow[ur, shift right, "\mu_2"'] \arrow[dr, shift right, "\pi_1"'] & & P_2 \arrow[ul, shift left, "\mu_1"] \arrow[dl, shift left, "\pi_2"] \\
& P_\emptyset \arrow[ul, shift right, "\mu_1"'] \arrow[ur, shift left, "\mu_2"] &
\end{tikzcd}$$

Figure 2. An inclusion-exclusion system for Q_2.

Our resolution of $K = \mathrm{Ker}(\pi_1) \cap \mathrm{Ker}(\pi_2)$ takes the form

$$0 \longrightarrow K \hookrightarrow P_{12} \xrightarrow{\partial_2} P_1 \oplus P_2 \xrightarrow{\partial_1} P_\emptyset \longrightarrow 0 \tag{18}$$

where the maps are

$$\partial_2 f = (-\pi_2(f), \pi_1(f)),$$

and

$$\partial_1(f_1, f_2) = \pi_1(f_1) + \pi_2(f_2).$$

Exactness at P_{12} is the statement that $K = \mathrm{Ker}(\partial_2)$, which is obvious. Exactness at $P_\emptyset$ is trivial because the π_j are surjective. As we see, even in this minimal case, exactness at $\mathbf{P}_1$ is both intriguing and nontrivial.

Remark 9.1. *Exactness of (18) at $\mathbf{P}_1 = P_1 \oplus P_2$ is equivalent to the statement that the fiber product $P_1 \times_{P_\emptyset} P_2$ is isomorphic to P_{12}/K.*

The fact that (18) is exact at $\mathbf{P}_1$ is a special case of Theorem 5.3, but we give an explicit formula:

Proposition 9.2. *Let $f_1 \in P_1$ and $f_2 \in P_2$ be such that $\pi_1 f_1 = \pi_2 f_2$. Then the element*

$$g = \mu_1 f_2 + \mu_2 f_1 - \mu_1\mu_2\pi_2 f_2 \qquad \in P_{12}$$

satisfies $\pi_1 g = f_2$ and $\pi_2 g = f_1$.

Proof. We first assume $\mathrm{char}\, F \neq 2$. By Proposition 7.2, $\partial_2\theta_1 + \theta_0\partial_1 = s_1 + 1$, so multiplying by s_1 from the right, using the fact that $s_1^2 = s_1$ (satisfying the polynomial $h_1(\lambda) = \lambda^2 - \lambda$), we obtain $1 = \partial_2(\theta_1 - \frac{1}{2}\theta_1 s_1) + \theta_0(\partial_1 - \frac{1}{2}\partial_1 s_1)$; namely, for every $f \in \mathbf{P}_1$ we have that $f = \partial_2(\theta_1 - \frac{1}{2}\theta_1 s_1)f + \theta_0(\partial_1 - \frac{1}{2}\partial_1 s_1)f$. Since $\partial_1 \circ s_1 = \partial_1$ by (16) (as $s_0 = 0$), when $\partial_1 f = 0$ we have that $f = \partial_2 g$ for

$$g = (\theta_1 - \frac{1}{2}\theta_1 s_1)f.$$

In order to express this formula in terms of the π_j and μ_j, let us recall the maps $s_1 : P_1 \oplus P_2 \to P_1 \oplus P_2$ of (14) and $\theta_1 : P_1 \oplus P_2 \to P_{12}$ of (17), defined by

$$s_1(h_1, h_2) = (\mu_1\pi_1 h_1, \mu_2\pi_2 h_2)$$

and

$$\theta_1(h_1, h_2) = \mu_1 h_2 - \mu_2 h_1.$$

(We also have $\theta_0 : P_\emptyset \to \mathbf{P}_1$ by $\theta_0 h = (\mu_1 h, \mu_2 h)$, which will not be necessary here). Write $f = (-f_1, f_2) \in \mathbf{P}_1$. Thus

$$\begin{aligned}
g &= (\theta_1 - \frac{1}{2}\theta_1 s_1)(-f_1, f_2) \\
&= \theta_1(-f_1, f_2) - \frac{1}{2}\theta_1 s_1(-f_1, f_2) \\
&= (\mu_1 f_2 - \frac{1}{2}\mu_1\mu_2\pi_2 f_2) + (\mu_2 f_1 - \frac{1}{2}\mu_2\mu_1\pi_1 f_1) \\
&= \mu_1 f_2 + \mu_2 f_1 - \frac{1}{2}(\mu_1\mu_2 + \mu_2\mu_1)\pi_1 f_1.
\end{aligned}$$

This element, although symmetric under switching indices, is not defined when $\mathrm{char}\, F = 2$. But $\pi_2[\mu_1, \mu_2] = [\pi_2, \mu_1]\mu_2 + [\mu_1, \pi_2\mu_2] = 0\mu_2 + [\mu_1, 1] = 0$, and by symmetry $\pi_1[\mu_1, \mu_2] = 0$ as well, so that $\mathrm{Im}([\mu_1, \mu_2]) \subseteq K$. We may therefore replace g by

$$g' = g - \frac{1}{2}[\mu_1, \mu_2]\pi_1 f_1 = \mu_1 f_2 + \mu_2 f_1 - \mu_1\mu_2\pi_1 f_1, \tag{19}$$

and this element is defined in any characteristic. We omit the verification that $\pi_1 g' = f_2$ and $\pi_2 g' = f_1$ as this is done in general in Theorem 10.1.

□

We thus have improved Theorem 5.3 when $n = 2$:

Corollary 9.3. *For $n = 2$, the complex (11) is exact over any field F.*

10. Common ancestors

Fix $1 \leq k < n$. We prove another curious property of inclusion-exclusion systems, which generalizes the exactness of (11) at $\mathbf{P}_{n-1}$ (but has nothing to do with exactness at $\mathbf{P}_{n-k}$, as may seem at first sight).

Let $(\{P_S\}, \{\pi_j, \mu_j\})$ be an inclusion-exclusion system of order n (Definition 3.1). Recall the definition of π_S and μ_S from (8)–(9), and as before, denote by $\bar{S}$ the complement of an index set S in $\{1, \dots, n\}$. Recall that by definition each component of f is an alternating function of its indices. To avoid alternating signs, in this section we define f_S to be $f_{i_1,\dots,i_r}$ where $i_1 < \cdots < i_r$ are the indices composing S, and write f^S for $f_{\bar{S}}$. We say that $g \in \mathbf{P}_n = P_{\{1,\dots,n\}}$ is a **common ancestor** of $f \in \mathbf{P}_{n-k}$ if $f^S = \pi_S g$ for every ordered set S of cardinality k. Namely, all the entries of f are determined by a single element at the top level. We say that an element $f \in \mathbf{P}_{n-k}$ is **coherent** if

$$\pi_i(f^{S\cup\{j\}}) = \pi_j(f^{S\cup\{i\}})$$

for every S of cardinality $k-1$, and every $i, j \notin S$. (Equivalently if $\pi_{S-S'}(f^{S'}) = \pi_{S'-S}(f^S)$ for every S, S' of cardinality k).

Theorem 10.1. *An element $f \in \mathbf{P}_{n-k}$ has a common ancestor if and only if it is coherent.*

Proof. Coherence is clearly necessary, because if g is a common ancestor then

$$\pi_i(f^{S\cup\{j\}}) = \pi_i\pi_{S\cup\{j\}}g = \pi_{S\cup\{i,j\}}g = \pi_j\pi_{S\cup\{i\}}g = \pi_j(f^{S\cup\{i\}}).$$

Sufficiency follows from the explicit formula for a common ancestor, given in Proposition 10.3 below. □

Lemma 10.2. *Let $k \geq 1$ and $m \geq 0$ be integers. Then*

$$\sum_i (-1)^i \binom{k}{i}\binom{m+i-1}{k-1} = (-1)^k \delta_{m,0}$$

(summing where both binomial coefficients are standardly defined).

Proof. Consider the sums $u(x) = \sum_{i=0}^{k} \binom{k}{i} x^i = (1+x)^k$ and

$$v(y) = \sum_{j=k-m}^{\infty} \binom{m+j-1}{k-1} y^j = y^{k-m} \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} y^j = y^{k-m}(1-y)^{-k},$$

using [10, Eq. (2.5.7)]. On one hand, the coefficient of x^0 in the Laurent series of

$$u(-x)v(x^{-1}) = \sum_{i=0}^{k} \sum_{j=k-m}^{\infty} (-1)^i \binom{k}{i} \binom{m+j-1}{k-1} x^{i-j}$$

is our sum $\sum_i (-1)^i \binom{k}{i} \binom{m+i-1}{k-1}$. On the other hand,

$$u(-x)v(x^{-1}) = (1-x)^k x^{m-k} (1-x^{-1})^{-k} = (-1)^k x^m;$$

so the coefficient of x^0 is $(-1)^k \delta_{m,0}$. $\square$

We can now give a formula for the common ancestor, providing an explicit proof for Theorem 10.1.

Proposition 10.3. *Let $f = \sum_{|C|=k} f^C$ be a coherent element of $\mathbf{P}_{n-k}$. For every index set S of cardinality $\geq k$, choose some subset $C_S \subseteq S$ of cardinality k. The element*

$$g = \sum_{|S|\geq k} (-1)^{|S|-k} \binom{|S|-1}{k-1} \mu_S \pi_{S-C_S} f^{C_S} \quad \in \mathbf{P}_n \tag{20}$$

is a well defined common ancestor for f.

For $k = 1$ we obtain a classical inclusion-exclusion formula

$$g = \sum_{|S|\geq 1} (-1)^{|S|-1} \mu_S \pi_{S-C_S} f^{C_S},$$

which specializes to (19) when $n = 2$. One way to obtain (20) is by induction on k, climbing from a coherent element of $\mathbf{P}_{n-k}$ to a coherent element of $\mathbf{P}_{n-k+1}$ etc.

Proof. Let S be an index set of cardinality $\geq k$. We first show that $\pi_{S-C_S} f^{C_S}$ is independent of $C_S \subseteq S$ (as long as $|C_S| = k$). Suppose $C, C' \subseteq S$ are subsets of cardinality k of S, with $C_0 = C \cap C'$ of cardinality $k-1$. Srite $C = C_0 \cup \{i\}$ and $C' = C_0 \cup \{i'\}$. Then

$$\begin{aligned} \pi_{S-C} f^C &= \pi_{S-C_0-\{i\}} f^{C_0\cup\{i\}} \\ &= \pi_{S-C_0-\{i,i'\}} \pi_{i'} f^{C_0\cup\{i\}} = \pi_{S-C_0-\{i',i\}} \pi_i f^{C_0\cup\{i'\}} \\ &= \pi_{S-C_0-\{i'\}} f^{C_0\cup\{i'\}} = \pi_{S-C'} f^{C'}; \end{aligned}$$

and this proves the claim because every two subsets of cardinality k can be connected by changes of one index at a time.

Fix an index set C of cardinality k. For every S, we choose C_S so that it contains $C \cap S$. We rearrange the sum by the intersection of S with C, and then replace all C_S by C:

$$\begin{aligned}
\pi_C g &= \sum_{|S|\geq k} (-1)^{|S|-k}\binom{|S|-1}{k-1}\pi_C\mu_S\pi_{S-C_S}f^{C_S} \\
&= \sum_{C_0\subseteq C}\ \sum_{|S|\geq k,\, S\cap C=C_0} (-1)^{|S|-k}\binom{|S|-1}{k-1}\pi_C\mu_S\pi_{S-C_S}f^{C_S} \\
&= \sum_{C_0\subseteq C}\ \sum_{|S|\geq k,\, S\cap C=C_0} (-1)^{|S|-k}\binom{|S|-1}{k-1}\mu_{S-C_0}\pi_{C-C_0}\pi_{S-C_S}f^{C_S} \\
&= \sum_{C_0\subseteq C}\ \sum_{|S|\geq k,\, S\cap C=C_0} (-1)^{|S|-k}\binom{|S|-1}{k-1}\mu_{S-C_0}\pi_{S\cup(C-C_0)-C_S}f^{C_S} \\
&= \sum_{C_0\subseteq C}\ \sum_{|S|\geq k,\, S\cap C=C_0} (-1)^{|S|-k}\binom{|S|-1}{k-1}\mu_{S-C_0}\pi_{S-C_0}f^{C};
\end{aligned}$$

now replace S by $S\mathbin{\dot\cup} C_0$:

$$= \sum_{C_0\subseteq C}\ \sum_{|S|\geq k-|C_0|,\, S\cap C=\emptyset} (-1)^{|S|+|C_0|-k}\binom{|S|+|C_0|-1}{k-1}\mu_S\pi_S f^C$$

which depends on C_0 only through its cardinality:

$$\begin{aligned}
&= \sum_{i=0}^{k}\binom{k}{i}\sum_{|S|\geq k-i,\, S\cap C=\emptyset}(-1)^{|S|-k+i}\binom{|S|+i-1}{k-1}\mu_S\pi_S f^C \\
&= \sum_{S\cap C=\emptyset}\left(\sum_i (-1)^{|S|-k+i}\binom{k}{i}\binom{|S|+i-1}{k-1}\right)\mu_S\pi_S f^C
\end{aligned}$$

and we are done by Lemma 10.2:

$$= \sum_{S\cap C=\emptyset}\delta_{|S|,0}\mu_S\pi_S f^C \;=\; f^C.$$

$\square$

11. Exactness at $\mathbf{P}_{n-1}$

The case $k=1$ of Theorem 10.1 states that a coherent element of $\mathbf{P}_{n-1}$ has a common ancestor in $\mathbf{P}_n$. We use this to provide a characteristic-free proof that (11) is exact at $\mathbf{P}_{n-1}$.

Some alteration of the signs will simplify notation. For every $h\in\mathbf{P}_{n-1}$, we let $h^*\in\mathbf{P}_{n-1}$ denote the element whose entries are $h^*_{1,\ldots,\hat{t},\ldots,n}=(-1)^{t-1}h_{1,\ldots,\hat{t},\ldots,n}$. Two facts now follow from Proposition 5.5:

(a) for $g\in\mathbf{P}_n$, $(\partial_n g)^*_{1,\ldots,\hat{t},\ldots,n}=\pi_t(g_{1,\ldots,n})$;
(b) for $f\in\mathbf{P}_{n-1}$, $(-1)^{t'-t}(\partial_{n-1}f)_{1,\ldots,\hat{t},\ldots,\hat{t'},\ldots,n}=\pi_t(f^*_{1,\ldots,\hat{t'},\ldots,n})-\pi_{t'}(f^*_{1,\ldots,\hat{t},\ldots,n})$ for every $t<t'$.

Theorem 11.1. *The complex (11) is exact at $\mathbf{P}_{n-1}$ over any field F.*

Proof. Let $f \in \mathbf{P}_{n-1}$ be an element such that $\partial_{n-1} f = 0$. Write f^* as a sum $f^* = \sum f^i$ where each $f^i \in P_{1\dots,\hat{\imath},\dots,n}$. By (b), the assumption that $\partial_{n-1} f = 0$ gives $\pi_i f^{i'} = \pi_{i'} f^i$ for every i, i', so the conditions of Proposition 10.3 hold (for $k = 1$). Let $g \in \mathbf{P}_n$ be the element defined by (20), which is characteristic-free. By the proposition, $\pi_j g = f^j$ for each j, so (a) gives us that $(\partial_n g)^* = \sum_j (\partial_n g)^*_{1,\dots,\hat{\jmath},\dots,n} = \sum_j \pi_j(g_{1,\dots,n}) = \sum f^j = f^*$, proving that $\partial_n g = f$. □

Remark 11.2. *We do not know if (11) is exact over fields of positive characteristic.*

12. Applications

We briefly present some applications of Theorem 10.1 to polynomials and permutations.

12.1. **Polynomials.** We now specialize Theorem 10.1 to the inclusion-exclusion system of Example 3.4. Recall that for any index set $S = \{i_1, \dots, i_k\}$, P_S is the space of multilinear polynomials in the letters $\{x_{i_1}, \dots, x_{i_k}\} \cup Y$, where Y is a fixed set. Fix $k < n$. A system of polynomials $\{f_S \in P_S : |S| = k\}$ is **coherent** if for every S of cardinality $k-1$, and every $i, i' \notin S$, we have that $(f_{S\cup\{i\}})|_{x_i=1} = (f_{S\cup\{i'\}})|_{x_{i'}=1}$. A polynomial $g \in P_{\{1,\dots,n\}}$ is a **common ancestor** if for every subset S of cardinality k, $g|_{x_i \mapsto 1, i \notin S} = f_S$.

Corollary 12.1. *Every coherent system of polynomials $\{f_S \in P_S : |S| = k\}$ has a common ancestor.*

Corollary 12.2. *The space of coherent systems of polynomials (a subspace of $\mathbf{P}_k = \bigoplus_{|S|=k} P_S$) is spanned by coherent systems of monomials.*

Indeed, the space $\mathbf{P}_n$ of common ancestors is spanned by monomials.

12.2. **Permutations.** Let us rephrase Example 3.4 in the language of permutations, by introducing an inclusion-exclusion system on permutations of subsets of $\{1, \dots, n\}$. We view permutations on a set $\{i_1, \dots, i_k\}$ as words with k distinct letter, where $1 \le i_1 < \cdots < i_k \le n$. For an index set S, let Sym_S denote the symmetric group over S (more commonly denoted S_S).

Example 12.3. *For an index set S, let $P_S = F[\mathrm{Sym}_S]$ be the group algebra of Sym_S. The map π_j is defined on $F[\mathrm{Sym}_S]$ (for $j \in S$) by deleting j from each permutation; and the map μ_j is defined on $F[\mathrm{Sym}_S]$ (for $j \notin S$) by concatenating j at the end of each permutation.*

A system of elements in the group algebras $f_S \in F[\mathrm{Sym}_S]$, ranging over index sets of cardinality k, is **coherent** if the **shadows**, obtained from cancellation of the indices in $S - S'$ in f_S and cancellation of the indices in $S' - S$ in $f_{S'}$, coincide as the same element of $F[\mathrm{Sym}_{S \cap S'}]$.

Corollary 12.4. *Every coherent system of elements $\{f_S \in F[\mathrm{Sym}_S] : |S| = k\}$ has a common ancestor in $F[\mathrm{Sym}_{\{1,\dots,n\}}]$.*

Of particular interest are distribution elements in $F[\mathrm{Sym}_S]$, namely, working over $\mathbb{R}$, elements with positive coefficients summing to 1, so that they naturally represent a distribution space. For example, when $k = 2$, every system of distribution elements is coherent, because every cancellations give the trivial distribution on a single element. A coherent system of distributions is **realizable** if it has a common ancestor which is a distribution element. By Corollary 12.4, every coherent system of distribution elements has a common ancestor. However, a coherent system of distributions need not be realizable:

Proposition 12.5. *The shadows of a distribution element in* $\mathrm{Sym}_{\{1,2,3\}}$ *satisfy* $1 \le \Pr\{12\} + \Pr\{23\} + \Pr\{31\} \le 2$.

Proof. Indeed, under any distribution on $\{1, 2, 3\}$, $\Pr\{ij\} = \Pr\{ijk\} + \Pr\{ikj\} + \Pr\{kij\}$ for any permutation i, j, k of $1, 2, 3$, and so $\Pr\{12\} + \Pr\{23\} + \Pr\{31\} = 1 + \Pr\{123\} + \Pr\{231\} + \Pr\{312\}$. □

Set $q = 1 - p$. We obtain as an example for $n = 3$ that the coherent element

$$(p[12] + q[21],\, p[23] + q[32],\, p[31] + q[13]) \in \mathbf{P}_2$$

is not realizable unless $\frac{1}{3} \le p \le \frac{2}{3}$.

However, we can prove the following. Let $\Pi\colon \mathbf{P}_n \to \mathbf{P}_k$ be the map sending $f \in \mathbf{P}_n$ to the vector whose entries are the shadows of f. A coherent system of distributions in $\mathbf{P}_k$ is realizable if and only if it is the image of a distribution element under Π.

Proposition 12.6. *The map* Π*, restricted to distribution elements in* $\mathbf{P}_n$*, is an open map.*

Proof. Let $v \in \mathbf{P}_n$ be an element in the interior of the closed simplex of distribution elements. Let $\Pi(v) + p \in \mathbf{P}_k$ be a coherent element. Then $p \in \mathbf{P}_k$ is coherent as well, and its coefficients sum to zero in each component. The map Π is onto the coherent subspace of $\mathbf{P}_k$ by Theorem 10.1. Indeed, an inverse map is given by (20), and its image has zero sum coefficients as well, because the sum of coefficients is preserved under deletion of entries. Let $g \in \mathbf{P}_n$ be a common ancestor for p. If p is small enough, then the ℓ_1-norm of g is smaller than the distance of v from the walls of the simplex, making $v+g$ a common ancestor of $\Pi(v)+p$, which is a distribution element. □

As an application let u_k be the element in $\mathbf{P}_k = \bigoplus_{|S|=k} F[\mathrm{Sym}_S]$ whose components are all uniform distributions. This element is clearly realizable, being composed of the shadows of the uniform distribution on $\{1, \dots, n\}$.

Corollary 12.7. *There is an open ball around* u_k *in the space of coherent systems, all of whose elements are realizable.*

Proof. Apply Proposition 12.6 to the image of the uniform distribution u_n. □

13. The absolute kernel

In this final section we give formulas for the dimension of the absolute kernel in an inclusion-exclusion system, equivalently a representation of $\mathcal{F}_n$.

13.1. Dimension of the absolute kernel.

Proposition 13.1. *In any inclusion-exclusion system $(\{P_S\}, \{\pi_j, \mu_j\})$, the absolute kernel $K = \bigcap \mathrm{Ker}(\pi_j)$ has dimension*

$$\dim(K) = \sum_{i=0}^{n} (-1)^i \dim(\mathbf{P}_{n-i}).$$

Proof. The Euler characteristic of the complex (11) is equal to the alternating sum of dimensions

$$\chi([\mathbf{P}]) = \dim(K) - \dim(\mathbf{P}_n) + \dim(\mathbf{P}_{n-1}) - \cdots + (-1)^{n-1} \dim(\mathbf{P}_0).$$

Since the complex is exact, we have that $\chi([\mathbf{P}]) = 0$, proving the claim. $\square$

13.2. **Spechtian polynomials.** A multilinear polynomial $f(x_1, \ldots, x_n; Y)$ in the variables $\{x_1, \ldots, x_n\} \cup Y$ is n**-Spechtian** if $f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n; Y) = 0$ for every i. The space of n-Spechtian polynomials, denoted $\mathrm{Sp}_{n,Y}$, is clearly the absolute kernel of the inclusion-exclusion system of Example 3.4. When $|Y| = m$ we denote $\mathrm{Sp}_{n,Y}$ as $\mathrm{Sp}_{n,m}$.

Spechtian polynomials are of special interest in PI-theory, see [5, Subsection 6.2]. We say that two polynomials are disjoint if they have no common variable. A higher commutator is, by definition, either a variable, or a commutator $[f_1, f_2]$ where f_1, f_2 are disjoint higher commutators. The Spechtian polynomials are sums of products of disjoint higher commutators [5, Proposition 6.2.1]. The polynomial identities of an algebra follow from its Spechtian identities [5, Corollary 6.2.2].

Corollary 13.2. *For any $n, m \geq 0$,*

(1) $\dim(\mathrm{Sp}_{n,m}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k+m)!$,
(2) $\dim(\mathrm{Sp}_{n,m})$ *is the number of permutations $\sigma \in S_{n+m}$ such that none of $1, \ldots, n$ is a fixed point.*
(3) $\dim(\mathrm{Sp}_{n,m}) \approx e^{-\frac{n}{n+m}} (n+m)!$ *when n/m is fixed and $n \to \infty$.*

Proof. In Example 3.4 we have that $\dim \mathbf{P}_k = \binom{n}{k}(n-k+m)!$, which gives the dimension of $\mathrm{Sp}_{n,m}$ by substitution in Proposition 13.1. The description as a number of permutations follows by standard inclusion-exclusion (on the conditions $\sigma(i) = i$ for $i = 1, \ldots, n$). The approximation follows because for a random permutation $\sigma \in S_{n+m}$, the number of fixed points in the range $\{1, \ldots, n\}$ has approximately the Poisson distribution with expectancy $\frac{n}{n+m}$. $\square$

Remark 13.3. *Taking $m = 0$ in Corollary 13.2 gives the known identity* $\dim(\mathrm{Sp}_{n,0}) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx e^{-1} n!$*, which is the number of derangements (=permutations without fixed points) of order n, as was proved in 1991, independently, by P. Donnelly,* *S. Kapoor and E. Reingold, and R. Phatarfod, see* [1]. *(Indeed, in the terminology of* [1], $\mathrm{Sp}_{n,0}$ *is the kernel of the Tsetlin adjacency matrix.)*

Another approach, based on polynomial identities, is to take R to be the free algebra in [4, Theorem 4.27(ii)], *so that the codimension series is* $\tilde{c}(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$*; then* $\sum_{n=0}^{\infty} \dim(\mathrm{Sp}_{n,0}) \frac{t^n}{n!} = \tilde{\gamma}(t) = e^{-t}\tilde{c}(t) = \frac{e^{-t}}{1-t}$.

13.3. **Dimension formula for $\mathcal{F}_n$.** In conclusion, we rephrase Proposition 13.1 in terms of a representation of the free left regular band $\mathcal{F}_n$. Putting Proposition 4.7 into Proposition 13.1, recalling that $\mathbf{P}_k = \bigoplus_{|S|=k} P_S$, and writing $\dim(\epsilon V) = \dim(V) - \dim(\mathrm{Ker}(\epsilon))$ for every element ϵ, we obtain:

Theorem 13.4. *Let V be a finite dimensional representation space of $\mathcal{F}_n$, generated by endomorphisms ϵ_j. The absolute kernel has dimension*

$$\dim(\bigcap_j \mathrm{Ker}(\epsilon_j)) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1 < \cdots < i_k} \dim(\mathrm{Ker}(\epsilon_{i_1} \cdots \epsilon_{i_k})).$$

To illustrate Theorem 13.4, we extract an amusing linear algebra exercise, luring first year students into the rudiments of left regular bands.

Corollary 13.5. *Let $\epsilon_1, \epsilon_2 : V \to V$ be idempotent endomorphisms on a finite dimensional vector space V. Assume $\epsilon_1\epsilon_2\epsilon_1 = \epsilon_1\epsilon_2$ and $\epsilon_2\epsilon_1\epsilon_2 = \epsilon_2\epsilon_1$. Then*

$$\dim(\mathrm{Ker}(\epsilon_1) \cap \mathrm{Ker}(\epsilon_2)) = \dim(\mathrm{Ker}(\epsilon_1)) + \dim(\mathrm{Ker}(\epsilon_2)) - \dim(\mathrm{Ker}(\epsilon_1\epsilon_2)).$$

Proof. This is the case $n = 2$ of Corollary 13.4. We also give a direct proof. Clearly $\mathrm{Ker}(\epsilon_2) \subseteq \mathrm{Ker}(\epsilon_1\epsilon_2)$, and $\mathrm{Ker}(\epsilon_1) \subseteq \mathrm{Ker}(\epsilon_1\epsilon_2\epsilon_1) = \mathrm{Ker}(\epsilon_1\epsilon_2)$. On the other hand $x - \epsilon_1\epsilon_2 x = (1-\epsilon_1)x + \epsilon_1(1-\epsilon_2)x \in \mathrm{Ker}(\epsilon_1) + \mathrm{Ker}(\epsilon_2)$ using the identity $\epsilon_2\epsilon_1\epsilon_2 = \epsilon_2\epsilon_1$, proving that

$$\mathrm{Ker}(\epsilon_1\epsilon_2) = \mathrm{Ker}(\epsilon_1) + \mathrm{Ker}(\epsilon_2).$$

We now have that

$$\begin{aligned} \dim(\mathrm{Ker}(\epsilon_1) \cap \mathrm{Ker}(\epsilon_2)) &= \dim(\mathrm{Ker}(\epsilon_1)) + \dim(\mathrm{Ker}(\epsilon_2)) - \dim(\mathrm{Ker}(\epsilon_1) + \mathrm{Ker}(\epsilon_2)) \\ &= \dim(\mathrm{Ker}(\epsilon_1)) + \dim(\mathrm{Ker}(\epsilon_2)) - \dim(\mathrm{Ker}(\epsilon_1\epsilon_2)). \end{aligned}$$

which proves the claim. □

13.4. **The symmetric case.** In the symmetric case of Remark 5.4, where the representation is equivariant with respect to an action of S_n, we get:

Corollary 13.6. *Let V be a representation as in Theorem 13.4, which is symmetric with respect to an S_n-action. Then*

$$\dim(\bigcap_j \mathrm{Ker}(\epsilon_j)) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \dim(\mathrm{Ker}(\epsilon_1 \cdots \epsilon_k)).$$

Corollary 13.7. *In Corollary 13.6, assume $n = p$ is prime. Then*

$$\dim(\bigcap_j \mathrm{Ker}(\epsilon_j)) \equiv \dim(\mathrm{Ker}(\epsilon_1 \cdots \epsilon_p)) \pmod{p}.$$

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