ON LOCAL CLUB CONDENSATION

GABRIEL FERNANDES

ABSTRACT. We obtain results on the condensation principle called local club condensation. We prove that in extender models an equivalence between the failure of local club condensation and subcompact cardinals holds. This gives a characterization of $\square_\kappa$ in terms of local club condensation in extender models. Assuming GCH, given an interval of ordinals I we verify that iterating the forcing defined by Holy-Welch-Wu, we can preserve GCH, cardinals and cofinalities and obtain a model where local club condensation holds for every ordinal in I modulo those ordinals which cardinality is a singular cardinal. We prove that if κ is a regular cardinal in an interval I, the above iteration provides enough condensation for the combinatorial principle $\mathrm{Dl}^*_S(\Pi^1_2)$, and in particular $\diamondsuit(S)$, to hold for any stationary $S \subseteq \kappa$.

1. INTRODUCTION

Local club condensation is a condensation principle that abstracts some of the condensation properties of L, Gödels constructible hierarchy. Local club condensation was first defined in [FH11] and it is part of the outer model program which searches for forcing models that have L-like features.

Convention 1.1. The class of ordinals is denoted by OR. The transitive closure of a set X is denoted by $\mathrm{trcl}(X)$, and the Mostowski collapse of a structure $\mathfrak{B}$ is denoted by $\mathrm{clps}(\mathfrak{B})$.

In order to define condensation principles we define filtrations which is an abstraction of the stratfication $\langle L_\alpha \mid \alpha < \mathrm{OR}\rangle$ of L.

Definition 1.2. Given ordinals $\alpha < \beta$ we say that $\langle M_\xi \mid \alpha < \xi < \beta\rangle$ is a *filtration* iff

(1) for all $\xi \in (\alpha, \beta)$, M_ξ is transitive, $\xi \subseteq M_\xi$,
(2) for all $\xi \in (\alpha, \beta)$, $M_\xi \cap \mathrm{OR} = \xi$,
(3) for all $\xi \in (\alpha, \beta)$, $|M_\xi| \leq \max\{\aleph_0, |\xi|\}$,
(4) if $\xi < \zeta$, then $M_\xi \subseteq M_\zeta$,
(5) if ξ is a limit ordinal, then $M_\xi = \bigcup_{\alpha<\xi} M_\alpha$.

Convention 1.3. Given a filtration $\langle M_\xi \mid \xi < \beta\rangle$, if β is a limit ordinal we let $M_\beta := \bigcup_{\gamma<\beta} M_\gamma$.

The following is an abstract formulation of the Condensation lemma that holds for the constructible hirerarchy $\langle L_\alpha \mid \alpha \in \mathrm{OR}\rangle$:

Definition 1.4. Suppose that κ and λ are regular cardinals and that $\vec{M} = \langle M_\alpha \mid \kappa < \alpha < \lambda\rangle$ is a filtration. We say that $\vec{M}$ satisfies *strong condensation* iff for every $\alpha \in (\kappa, \lambda)$ and every $(X, \in) \prec_{\Sigma_1} (M_\alpha, \in)$ there exists $\bar{\alpha}$ such that $\mathrm{clps}(X, \in) = (M_{\bar{\alpha}}, \in)$.

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While strong condensation is not consistent with the existence of large cardinals, see [FH11] and [SV04], Local club condensation, which we define below, is consistent with any large cardinal, see [FH11, Theorem1].

Definition 1.5 (Holy,Welch,Wu,Friedman [HWW15],[FH11])**.** Let κ be a cardinal of uncountable cofinality. We say that $\vec{M} = \langle M_\beta \mid \beta < \kappa \rangle$ is a witnesses to the fact that *local club condensation holds in* (η, ζ), and denote this by $\langle H_\kappa, \in, \vec{M} \rangle \models \mathsf{LCC}(\eta, \zeta)$, iff all of the following hold true:

1. $\eta < \zeta \leq \kappa + 1$;
2. $\vec{M}$ is a *filtration* such that $M_\kappa = H_\kappa$ [1],
3. For every ordinal α in the interval (η, ζ) and every sequence $\mathcal{F} = \langle (F_n, k_n) \mid n \in \omega \rangle$ such that, for all $n \in \omega$, $k_n \in \omega$ and $F_n \subseteq (M_\alpha)^{k_n}$, there is a sequence $\vec{\mathfrak{B}} = \langle \mathcal{B}_\beta \mid \beta < |\alpha| \rangle$ having the following properties:
   (a) for all $\beta < |\alpha|$, $\mathcal{B}_\beta$ is of the form $\langle B_\beta, \in, \vec{M} \restriction (B_\beta \cap \mathrm{OR}), (F_n \cap (B_\beta)^{k_n})_{n \in \omega} \rangle$;
   (b) for all $\beta < |\alpha|$, $\mathcal{B}_\beta \prec \langle M_\alpha, \in, \vec{M} \restriction \alpha, (F_n)_{n \in \omega} \rangle$;[2]
   (c) for all $\beta < |\alpha|$, $\beta \subseteq B_\beta$ and $|B_\beta| < |\alpha|$;
   (d) for all $\beta < |\alpha|$, there exists $\bar{\beta} < \kappa$ such that
   $$\mathrm{clps}(\langle B_\beta, \in, \langle B_\delta \mid \delta \in B_\beta \cap \mathrm{OR} \rangle \rangle) = \langle M_{\bar{\beta}}, \in, \langle M_\delta \mid \delta \in \bar{\beta} \rangle \rangle;$$
   (e) $\langle B_\beta \mid \beta < |\alpha| \rangle$ is $\subseteq$-increasing, continuous and converging to M_α.

For $\vec{\mathfrak{B}}$ as in Clause (3) above we say that $\vec{\mathfrak{B}}$ *witnesses* $\mathsf{LCC}(\eta, \zeta)$ *at* α *with respect to* $\mathcal{F}$. We write $\mathsf{LCC}(\eta, \zeta]$ for $\mathsf{LCC}(\eta, \zeta + 1)$.

In section 2 we present our resutls regarding Local Club Condensation in extender models. An *extender model* is an inner model of the form $L[E]$, it is a generalization of L that can accommodate large cardinals. An inner model of the form $L[E]$ is the smallest transitive proper class that is a model of ZF and is closed under the operator $x \mapsto x \cap E$, where $E : \mathrm{OR} \to V$ and each $E_\alpha = \emptyset$ or E_α is a partial extender. $L[E]$ models can be stratified using the L-hirearchy and the J-hirearchy, for example:

- $J^E = \emptyset$,
- $J^E_{\alpha+1} = rud_E(J^E_\alpha \cup \{J^E_\alpha\})$,
- $J^E_\gamma = \bigcup_{\beta < \gamma} J^E_\beta$ if γ is a limit ordinal.

and finally
$$L[E] = \bigcup_{\alpha \in \mathrm{OR}} J^E_\alpha.$$

In [FH11, Theorem 8] it is shown that Local Club Condensation holds in various extender models, we extend [FH11, Theorem 8] to an optimal result for extender models that are weakly iterable (see Defnition 2.7). We carachterize Local club condensation in extender models in terms of subcompact cardinals[3].

Theorem A. *If $L[E]$ is an extender model that is weakly iterable, then given an infinite cardinal κ the following are equivalent:*

(a) $\langle L_{\kappa^+}[E], \in, \langle L_\beta[E] \mid \beta \in \kappa^+ \rangle \rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++}]$.
(b) $L[E] \models$ (κ *is not a subcompact cardinal*).

In addtion for every limit cardinal κ with $\mathrm{cf}(\kappa) > \omega$ we have
$$\langle L_{\kappa^+}[E], \in, \langle L_\beta[E] \mid \beta \in \kappa^+ \rangle \rangle \models \mathsf{LCC}(\kappa, \kappa^+].$$

[1] See Convention 1.3

[2] Note that the case $\alpha = \kappa$ uses Convention 1.3.

[3] A subcompact cardinal is a large cardinal that is located in the consistency strengh hirearchy below a supercompact cardinal and above a superstrong cardinal. See definition in [SZ01a]

We warn the reader that it is not known how to construct an extender model that is weakly iterable and has a subcompact cardinal, but this is part of the aim of the inner model theory program and it is desirable to know what hold in such models.

Corollary A provides an equivalence between $\square_\kappa$ and a condensation principle that holds in the interval (κ^+, κ^{++}), Corollary A is immediate from Theorem A and the main result in [SZ04]:

Corollary A. *If $L[E]$ is an extender model with Jensen's λ-indexing that is weakly iterable, then given κ, an $L[E]$ cardinal, the following are equivalent:*

(a) $L[E] \models \square_\kappa$
(b) $\langle L_\kappa[E], \in, \langle L_\beta[E] \mid \beta < \kappa^+\rangle\rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++})$

We verify that a subcompact cardinal is an even more severe impediment for LCC to hold:

Theorem B. *Suppose $L[E]$ is an extender model with Jensen's λ-indexing such that every countable elementary submodel of $L[E]$ is (ω_1+1, ω_1)-iterable. In $L[E]$, if an ordinal κ is a subcompact cardinal, then there is no $\vec{M}$ such that $\langle M_{\kappa^{++}}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++})$ and $M_{\kappa^{++}} = H^{L[E]}_{\kappa^{++}}$ and $M_{\kappa^+} = H^{L[E]}_{\kappa^+}$.*

In section 3 we prove how to force local club condensation on a given interval of ordinals I modulo ordinals with singular cardinality (Theorem C) . It was already obtained in [FH11] a model where local club condensation holds on arbitrary intervals I including ordinals with singular cardinality, this was done via class forcing, although we do not obtain as much condensation as in [FH11], building on [HWW15] we define a set forcing $\mathbb{P}$ which is simpler than the forcing in [FH11], and which will force enough condensation for a few applications, see section 4.

Theorem C. *If GCH holds and κ is a regular cardinal and α is an ordinal, then there is a set forcing $\mathbb{P}$ which is $<\kappa$-directed closed and $\kappa^{+\alpha+1}$-cc, GCH preserving such that in $V^{\mathbb{P}}$ there is a filtration $\langle M_\alpha \mid \alpha < \kappa^{+\alpha+1}\rangle$ such that for every regular cardinal $\theta \in [\kappa, \kappa^{+\alpha+1}]$ we have $H_\theta = M_\theta$ and $\langle M_\alpha \mid \alpha < \kappa^{+\alpha}\rangle \models \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\alpha})$.*

In Section 4 we show that the iteration of the forcing from [HWW15] implies $\mathrm{Dl}^*_S(\Pi^1_2)$ (see definition in [FMR20]) which is a combinatorial principle defined in [FMR20] and is a variation of Devlin's $\diamondsuit^\sharp_\kappa$ (see [Dev82]).

Corollary D. *Let κ be an uncountable regular cardinal and let μ be a cardinal such that $\mu^+ \leq \kappa$. If GCH holds, then there is a set forcing $\mathbb{P}$ which is $<\mu^+$-directed and κ^+-cc, GCH preserving, such that in $V^{\mathbb{P}}$ we have $\mathrm{Dl}^*_S(\Pi^1_2)$, in particular $\diamond(S)$, for any stationary $S \subseteq \kappa$.*

2. Local club condensation in extender models

The main result in this section is Theorem A which extends [FH11][Theorem 8] and gives a characterization of local club condensation in terms of subcompact cardinals. For the standard notation on inner model theory and fine structure like *premouse, projectum, standard parameter* and etc. we refer the reader to [Zem02].

Definition 2.1. Given a premouse $\mathcal{M}$, a parameter $p \in (\mathcal{M} \cap \mathrm{OR}^{<\omega})$ and $\xi \in \mathcal{M} \cap \mathrm{OR}$ and $\langle \varphi_i \mid i \in \omega\rangle$ a primitive recursive enumeration of all Σ_1 formulas in the premice language we define

$$T^M_p(\xi) = \{(a, i) \in (\xi^{<\omega} \times \omega) \mid M \models \varphi_i(a, p)\}.$$

Fact 2.2. *Given $\langle \varphi_n \mid n \in \omega\rangle$ a primitive recursive enumeration of all Σ_1 formulas in the premice language, there exists a Σ_1-formula $\Phi(w, x, y)$ in the premice language such that for any premouse $\mathcal{M}$ the following hold:*

- *If $n \in \omega$ is such that $\varphi_n(x) = \exists y \phi_n(x, y)$ and ϕ_n is Σ_0, then for every $x \in \mathcal{M}$ there exists $y_0 \in \mathcal{M}$ such that $(\mathcal{M} \models \phi_n(x, y_0))$ iff there exists $y_1 \in \mathcal{M}$ such that $(\mathcal{M} \models \Phi(n, x, y_1))$*
- *For every $n \in \omega$ and for every $x \in \mathcal{M}$, if there are $y_0, y_1 \in \mathcal{M}$ such that $(\mathcal{M} \models \Phi(n, x, y_0) \wedge \Phi(n, x, y_1)))$ then $y_0 = y_1$*

Definition 2.3. Let $\mathcal{M}$ be a premouse we denote by $h_1^{\mathcal{M}}$ the partial function from $\omega \times \mathcal{M}$ into $\mathcal{M}$ defined by the formula Φ from Fact 2.2. Given $X \subseteq \mathcal{M}$ and $p \in \mathcal{M}$ we denote by $h_1^{\mathcal{M}}[X, p]$ the set $h_1^{\mathcal{M}}[(X \times \{p\})^{<\omega}]$.

Fact 2.4. *(1) Suppose $L[E]$ is an extender model. Let γ be an ordinal such that $E_\gamma \neq \emptyset$, then there exists $g \in J^E_{\gamma+1}$ such that $g : \lambda(E_\gamma) \to \gamma$ onto.*
(2) $\mathcal{P}(J^E_\gamma) \cap J^E_{\gamma+1} = \Sigma_\omega(J^E_\gamma)$

Lemma 2.5. *Suppose $L[E]$ is an extender model and γ is such that $E_\gamma \neq \emptyset$ and $L_\gamma[E] = J^E_\gamma$. Then there exists $g \in L_{\gamma+1}[E]$ such that $g : \lambda(E_\gamma) \to \gamma$ and g is onto.*

Proof. It follows from Fact 2.4 and that $\Sigma_\omega(J^E_\gamma) = L_{\gamma+1}[E]$. □

Remark 2.6. Notice that in particular for any premouse $\mathcal{M}$, if $\gamma \in \mathcal{M} \cap \mathrm{OR}$ and $\mathcal{M} \models$ "γ is a cardinal" it follows from Fact 2.4 that $E_\gamma = \emptyset$, as otherwise

$$J^E_{\gamma+1} \models \text{“}\gamma \text{ is not a cardinal,”}$$

and hence

$$\mathcal{M} \models \text{“}\gamma \text{ is not a cardinal.”}$$

Definition 2.7. We say that an extender model $L[E]$ is *weakly iterable* iff for every $\alpha \in \mathrm{OR}$ if there exists an elementary embedding $\pi : \bar{\mathcal{M}} \to (J^E_\alpha, \in, E|\alpha, E_\alpha)$, then $\bar{\mathcal{M}}$ is $(\omega_1 + 1, \omega_1)$-iterable.[4]

Lemma 2.8. *Let $L[E]$ be an extender model that is weakly iterable and let κ be a cardinal in $L[E]$. Suppose $i : \mathcal{N} \to \mathcal{M}$ is the inverse of the Mostowisk collapse of $h_1^{\mathcal{M}}[\gamma \cup \{p_1^{\mathcal{M}}\}]$, $\rho_1(N) = \gamma$, $\mathrm{crit}(\pi) = \gamma$, $\gamma < \kappa$, $\mathcal{M} = \langle J^E_\alpha, \in, E \restriction \alpha, E_\alpha \rangle$ for some $\alpha \in (\kappa^+, \kappa^{++})$. Then $\mathcal{N} \triangleleft \mathcal{M}$ if and only if $E_\gamma = \emptyset$.*

Proof. The proof is a special case of condensation lemma. Suppose that $E_\gamma = \emptyset$, we will verify that $\mathcal{N} \triangleleft \mathcal{M}$.

Let $H \prec_{\Sigma_\omega} V_\Omega$ for some Ω large enough, where $i \in H$ and H is countable. Let $\pi : \bar{H} \to V_\Omega$ be the inverse of the Mostowisk colapse of H, let $\pi(\bar{\mathcal{N}}) = \mathcal{N}$, $\pi(\bar{\mathcal{M}}) = \mathcal{M}$ and $\pi(\bar{i}) = i$.

Let e be an enumeration of $\bar{\mathcal{M}}$ and let Σ be an e-minimal $(\omega_1, \omega_1 + 1)$-strategy for $\bar{M}$ [5]. Since $\bar{i}$ embedds $\langle \bar{\mathcal{M}}, \bar{\mathcal{N}}, \bar{\gamma} \rangle$ into $\bar{\mathcal{M}}$, it follows from 9.2.12 in [Zem02] that we can compare $\langle \bar{\mathcal{M}}, \bar{\mathcal{N}}, \bar{\gamma} \rangle$ and $\bar{\mathcal{M}}$ and we have the following:

- $\bar{\mathcal{M}}$ wins the comparison,
- the last model on the phalanx side is above $\bar{\mathcal{N}}$,
- there is no drop on the branch of the phalanx side.

From the fact that $h_1^{\mathcal{N}}(\gamma \cup \bar{p}) = \mathcal{N}$ it follows that $h_1^{\bar{\mathcal{N}}}(\bar{\gamma} \cup q) = \bar{\mathcal{N}}$ where $\pi(q) = \bar{p}$. This implies that $\bar{\mathcal{N}}$ can not move in the comparison, as otherwise it would drop and we already know that it is the $\bar{\mathcal{M}}$ side which wins the comparison. Let $\mathcal{T}$ be the iteration tree on $\bar{\mathcal{M}}$ and $\mathcal{U}$ the iteration tree on the phalanx $\langle \bar{M}, \bar{N}, \bar{\gamma} \rangle$.

Claim 2.8.1. $\bar{\mathcal{N}} \neq \mathcal{M}^{\mathcal{T}}_\infty$

[4]See definiton 9.1.10 in [Zem02] for the definition of $(\omega_1 + 1, \omega_1)$-iterable.

[5]The existence of an e-minimal iteration strategy follows from the hypothesis that $L[E]$ is weakly iterable and Neeman-Steel lemma, see [Zem02, Theorem9.2.11]

Proof. We already know that $\bar{\mathcal{N}} \lhd \mathcal{M}^{\mathcal{T}}_\infty$. If $\bar{\mathcal{M}}$ does not move then $\bar{\mathcal{N}} \neq \mathcal{M}^{\mathcal{T}}_\infty = \mathcal{M}$ since they have different cardinality. Suppose $\mathcal{T}$ is non-trivial and $\mathcal{M}^{\mathcal{T}}_\infty = \bar{\mathcal{N}}$ let $b^{\mathcal{T}}$ be the main branch in $\mathcal{T}$. Let η be the last drop in $b^{\mathcal{T}}$. In order to $\mathcal{M}^{\mathcal{T}}_\infty$ be 1-sound we need $\mathrm{crit}(E^{\mathcal{T}}_\eta) < \rho_1(\mathcal{M}^{\mathcal{T}}_\eta)$ and since $\lambda(E^{\mathcal{T}}_0) > \gamma$ we have $\lambda(E^{\mathcal{T}}_\eta) > \gamma$. This implies that $\rho_1(\mathcal{M}^{\mathcal{T}}_\infty) \geq \rho_1(\mathcal{M}^{\mathcal{T}}_{\eta+1}) > \pi^{\mathcal{T}}_{\eta^*,\eta+1}(\kappa_\eta) \geq \gamma = \rho_1(\mathcal{N})$, which is a contradiction since we are assuming that $\mathcal{M}^{\mathcal{T}}_\infty = \bar{\mathcal{N}}$. □

Since $\bar{\mathcal{N}}$ is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_\infty$ it will follows that $\bar{\mathcal{M}}$ does not move. For a contradiction, suppose $\bar{\mathcal{M}}$ moves then the index $\lambda(E^{\mathcal{U}}_0)$ of the first extender used on the $\bar{\mathcal{M}}$ side is greater than γ since $\bar{\mathcal{N}} \restriction \gamma = \bar{\mathcal{M}} \restriction \gamma$ and, by our hypothesis, $E_\gamma = \emptyset$. Moreover the cardinal in $\mathcal{M}^{\mathcal{U}}_{lh(\mathcal{T}-1)}$, the last model in the iteration on the $\bar{\mathcal{M}}$ side of the comparison. We have the following:

- $\bar{\mathcal{N}}$ is a proper initial segment of $\mathcal{M}^{\mathcal{T}}_\infty$,
- $\lambda(E^{\mathcal{U}}_0) \leq (\bar{\mathcal{N}} \cap \mathrm{OR})$,
- $h^{\bar{\mathcal{N}}}_1(\bar{\gamma} \cup q) = \bar{\mathcal{N}}$,
- $h^{\bar{\mathcal{N}}}_1 \restriction (\bar{\gamma} \cup \{q\}) \in \mathcal{M}^{\mathcal{T}}_\infty$,

then there exists a surjection from $\bar{\gamma}$ onto the index of $E^{\mathcal{T}}_0$ in $\mathcal{M}^{\mathcal{T}}_\infty$ which is a contradiction.

Thus we must have $\bar{\mathcal{N}} \lhd \bar{\mathcal{M}}$ and by elementarity of π we have $\mathcal{N} \lhd \mathcal{M}$. □

Lemma 2.9. *Let $L[E]$ be an extender model that is weakly iterable. In $L[E]$, let κ be a cardinal which is not a subcompact cardinal. Let $\beta \in (\kappa^+, \kappa^{++})$ and $\mathcal{M} = (J^E_\beta, \in, E \restriction \beta, E_\beta)$ and suppose that $\rho_1(\mathcal{M}) = \kappa^+$. Then there is club $C \subseteq \kappa^+$ such that for all $\gamma \in C$ if $\mathcal{N} = clps(h^{\mathcal{M}}_1(\gamma \cup \{p^{\mathcal{M}}_1\}))$ then $\rho_1(\mathcal{N}) = \gamma$.*

Proof. Let g be a function with domain κ^+ such that for each $\xi < \kappa^+$ we have that $g(\xi) = h^{\mathcal{M}}_1(\xi \cup \{p^{\mathcal{M}}_1\}) \cap \mathrm{OR}^{<\omega}$.

Let $f : \kappa^+ \to \kappa^+$ where given $\gamma < \kappa^+$, $f(\gamma)$ is the least ordinal such that for every $r \in g(\xi)$ we have that $T^{\mathcal{M}}_r(\gamma) \in J^E_{f(\gamma)}$. Notice that $T^{\mathcal{M}}_r(\gamma) \subseteq \bigcup_{n\in\omega} \mathcal{P}([\gamma]^n)$, hence it can be codded as a subset of γ and therefore, by acceptability, it follows that $f(\gamma) < \kappa^+$.

Let C be a club subset of the club of closure points of f and such that $\gamma \in C$ implies $\gamma = h^{\mathcal{M}}_1(\gamma \cup \{p^{\mathcal{M}}_1\}) \cap \kappa^+$. We will verify that C is the club we sought.

Let $\gamma \in C$. Let $\pi : \mathcal{N} \to J^E_\beta$ be the inverse of the Mostowisk collapse of $h^{\mathcal{M}}_1(\gamma \cup \{p^{\mathcal{M}}_1\})$. Then for each $\xi < \gamma$ we have $T^{\mathcal{N}}_r(\xi) = T^{\mathcal{M}}_{\pi(r)}(\xi) \in J^E_\gamma = \mathcal{N} \restriction \gamma$. Therefore $\rho_1(\mathcal{N}) \geq \gamma$.

Notice that by a standard diagonal argument $a = \{\xi \in \gamma \mid \xi \notin h^{\mathcal{N}_\gamma}_1(\xi, p_1)\} \notin \mathcal{N}_\gamma$ since $\mathcal{N}_\gamma = h_1[\gamma \cup \{p_1\}]$, thus $\rho^{\mathcal{M}}_1 \geq \gamma$. □

Lemma 2.10. *Let $L[E]$ be an extender model that is weakly iterable. Given $\kappa \in \mathrm{OR}$ if κ is a successor cardinal in $L[E]$ then the following are equivalent:*

(a) $\langle L_{\kappa^+}[E], \in, \langle L_\beta[E] \mid \beta \in \kappa^+\rangle\rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++}]$.
(b) $L[E] \models$ (κ *is not a subcompact cardinal*).

and if κ is a limit cardinal of uncountable cofinality, then

$$\langle L_{\kappa^+}[E], \in, \langle L_\beta[E] \mid \beta \in \kappa^+\rangle\rangle \models \mathsf{LCC}(\kappa, \kappa^+].$$

Proof. Let $\alpha \in (\kappa^+, \kappa^{++})$. Let $\beta \geq \alpha$ such that $\beta \in (\kappa^+, \kappa^{++})$ and $\rho_1((J^E_\beta, \in, E \restriction \beta, E_\beta)) = \kappa^+$. Let $\mathcal{M} = (J^E_\beta, \in, E \restriction \beta, E_\beta)$,

$$D = \{\gamma < \kappa^+ \mid h_1(\gamma \cup \{p_1\}) \cap \kappa^+ = \gamma\}$$

and for each $\gamma \in D$ let $\mathcal{N}_\gamma = \mathrm{clps}(h_1(\gamma \cup \{p_1\}))$. By Lemma 2.9 D contains a club $F \subseteq D$ such that $\gamma \in F$ implies that there are $\pi_\gamma : N_\gamma \to J^E_\beta$ where π_γ is $\Sigma^{(1)}_1$,

$\pi_\gamma \restriction \gamma = id \restriction \gamma$, $\pi_\gamma(\gamma) = \kappa^+$, $\rho_1(N_\gamma) = \gamma$. We can also assume that $\gamma \in F$ implies $L_\gamma[E] = J^E_\gamma$.

We verify first the implication $\neg(b) \Rightarrow \neg(a)$.

Suppose that $\langle B_\gamma \mid \gamma < |\alpha|\rangle$ is a continuous chain of elementary submodels of $\mathcal{N} = \langle J^E_\alpha, \in, E|\alpha, E_\alpha\rangle$ such that for all $\gamma < |\alpha|$ we have $|B_\gamma| < |\alpha|$ and $\bigcup_{\gamma<|\alpha|} B_\gamma = \mathcal{M}$. We will verify that for stationary many γ's we have that $\mathrm{clps}(B_\gamma)$ is not of the form J^E_ζ for any ζ.

As $|\alpha| = \kappa$ is a regular cardinal it follows that for club many γ's we have $B_\gamma = h^{\mathcal{M}}_1(\gamma \cup \{p_1\}) \cap \mathcal{N} = \pi^{-1}(L_\alpha[E])$

From $\neg(b)$ by Schimmerling-Zeman carachterization of $\square_\kappa$ (see [SZ01a]][Theorem 0.1]), we can assume that for stationary many $\gamma \in F$ we have $E^{\mathcal{M}}_\gamma \neq \emptyset$. Notice that $N_\gamma \models$ "γ is a cardinal" and therefore $E^{\mathcal{N}_\gamma} = \emptyset$ by Proposition 2.5. On the other hand, from Proposition 2.5 we have $L^E_{\gamma+1} \models$ "γ is not a cardinal". Since $L_{\gamma+1}[E] \subseteq J_{\gamma+1}[E]$ it follows that $\mathcal{N}_\gamma = \mathrm{clps}(B_\gamma)$ is different from J^E_ζ for every $\zeta > \gamma$. Therefore $\mathsf{LCC}(\kappa^+, \kappa^{++})$ does not hold.

Next we verify $(b) \rightarrow (a)$. Suppose $\mathfrak{N} = \langle L^E_\alpha, \in, E|\alpha, E_\alpha, (\mathcal{F}_n \mid n \in \omega)\rangle$. We can assume without loss of generality that β is large enough so that $\mathfrak{N} \in \mathcal{M}$.

We verify that $\langle L_\tau[E] \mid \tau < \kappa^{++}\rangle$ witnesses $\mathsf{LCC}(\kappa^+, \kappa^{++}]$ at α. Let $\vec{\mathcal{R}} := \{h^{\mathcal{M}}_1[\gamma \cup \{p^{\mathcal{M}}_1\} \cup \{u^{\mathcal{M}}_1\}] \mid \gamma < \kappa^+\}$ where for any given $X \subseteq \mathcal{M}$, $h^{\mathcal{M}}_1[X]$ denotes the Σ_1-Skolem hull of X in $\mathcal{M}$ and $p^{\mathcal{M}_1}$ is the first standard parameter. It follows that

$$C = \{\gamma < \kappa^+ \mid \mathrm{crit}(\mathrm{clps}(h^{\mathcal{M}}_1[\gamma \cup \{p^{\mathcal{M}}_1, u^{\mathcal{M}}_1\}])) = \gamma\}$$

is a club and by lemma 2.9

$$D = \{\gamma < \kappa^+ \mid \rho_1(\mathcal{N}_\gamma) = \gamma\}$$

is also a club. From Theorem 1 in [SZ01b] and (b) it follows that there is a club $F \subseteq \{\gamma < \kappa^+ \mid E_\gamma = \emptyset\}$. By Lemma 2.8, for every $\gamma \in D \cap F$ we have $N_\gamma \lhd \mathcal{M}$. We have $L_\alpha[E] \lhd \mathcal{M}$, therefore $\mathrm{clps}(L_\alpha[E] \cap h^{\mathcal{M}}_1(\gamma \cup \{p_1\})) \lhd \mathcal{N}_\gamma$, hence $\mathrm{clps}(L_\alpha[E] \cap h^{\mathcal{M}}_1(\gamma \cup \{p_1\})) = \mathrm{clps}(B_\gamma) \lhd \mathcal{M}$ which verifies the equivalence between $(b) \rightarrow (a)$

Now suppose κ is a limit cardinal. The same argument used for the implication $(b) \Rightarrow (a)$ follows with the difference that we do not use Theorem 1 of [SZ01b], instead we use that the cardinals below κ form a club and that for every cardinal $\mu < \kappa$ we have $E_\mu = \emptyset$. □

Definition 2.11. Given two predicates A and E we say that A is equivalent to E iff $J^A_\alpha = J^E_\alpha$ for all $\alpha < \mathrm{OR}$.

Corollary 2.12. *If $A \subseteq \mathrm{OR}$ is such that*

- $L[A] \models (\kappa$ *is a subcompact cardinal* $)$, *and*
- $\langle L_{\kappa^{++}}[A], \in, \langle L_\beta[A] \mid \beta < \kappa^{++}\rangle\rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++}]$,

then there is no extender sequence such that $L[E]$ is weakly iterable and E is equivalent to A.

Remark 2.13. In [FH11] from the hypothesis that there is κ a subcompact cardinal in V, it is obtained $A \subseteq \mathrm{OR}$ in a class generic extension which satisfies the hypothesis of corollary 2.12.

Corollary 2.14. *Suppose that $L[E]$ is a extender model with Jensen's λ-indexing and for every ordinal α the premouse $\mathcal{J}^E_\alpha$ is weakly iterable. If κ is an ordinal such that*

$$L[E] \models \kappa \textit{ is a subcompact cardinal,}$$

then for no $\vec{M} = \langle M_\alpha \mid \alpha < \kappa^{++}\rangle$ with $M_{\kappa^+} = H_{\kappa^+}$, $M_{\kappa^{++}} = H_{\kappa^{++}}$ and $\langle H_{\kappa^{++}}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++})$.

Definition 2.15. We say that a nice filtration $\vec{M}$ for H_{κ^+} strongly fails to condensate iff there is a stationary set $S \subseteq \kappa^+$ such that for any $\beta \in S$ and any continuous chain $\vec{B}$ of elementary submodels of M_β there are stationary many points α where B_α does not condensate.

Lemma 2.16. *If $\vec{M}$ is a filtration for H_{κ^+} with $M_\kappa = H_\kappa$ that strongly fails to condensate, then there is no filtration $\vec{N}$ of H_{κ^+} with $N_\kappa = H_\kappa$ that witnesses $\mathsf{LCC}(\kappa, \kappa^+)$.*

Proof. Let $\vec{N}$ be a filtration of H_{κ^+} with $N_\kappa = H_\kappa$ and $N_{\kappa^+} = H_{\kappa^+}$. Then there is a club $D \subseteq \kappa^+$ where $N_\beta = M_\beta$ for every $\beta \in D$. Let $\beta \in S \cap D$, and let $\vec{\mathfrak{B}} = \langle \mathfrak{B}_\tau \mid \tau < |\beta| = \kappa\rangle$ be any chain of elementary submodels of $M_\beta = N_\beta$. □

Corollary 2.17. *Suppose V is an extender model which is weakly iterable. If there exists κ such that $L[E] \models$ "κ is a subcompact cardinal", then there is no sequence $\vec{M} = \langle M_\alpha \mid \alpha < \kappa^+\rangle$ in $L[E]$ such that $\langle M_{\kappa^+}, \in, \vec{M}\rangle \models \mathsf{LCC}(\kappa^+, \kappa^{++})$.*

3. Forcing Local Club Condensation

In [FH11] it is shown, via class forcing, how to obtain a model of local club condensation for all ordinals above ω_1. Later a simpler forcing was presented in [HWW15] which forces condensation on an interval of the form (κ, κ^+) where κ is a regular cardinal. In this section we show that iterating the forcing from [HWW15] and obtain a set forcing $\mathbb{P}$ which forces local club condensation on all ordinals of an interval $(\kappa, \kappa^{+\alpha})$ modulo ordinals with singular cardinality. We will denote by $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\alpha})$ (see Definition 3.2) the property that local club condensation holds for all ordinals in the interval $(\kappa, \kappa^{+\alpha})$ modulo those which cardinality is a singular cardinal.

Iterating the forcing from [HWW15] gives us a set forcing which is relatively simpler than the class forcing from [FH11] and $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\alpha})$ is enough condensation for applications where $\mathsf{LCC}(\kappa, \kappa^{+\alpha})$ was used before, see Section 4 for applications.

Definition 3.1. Let κ be a regular cardinal and α an ordinal such that $\kappa^{+\alpha}$ is a regular cardinal. We say that $\Psi(\vec{M}, \vec{\theta})$ holds iff $\vec{M} = \langle M_\gamma \mid \gamma < \kappa^{+\alpha}\rangle$ is a filtration and for every regular cardinal $\theta \in (\kappa, \kappa^{+\alpha})$ we have $M_\theta = H_\theta$.

Definition 3.2. Let κ be a regular cardinal and α an ordinal. We say that $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\alpha})$ holds iff there is a filtration $\vec{M} = \langle M_\gamma \mid \gamma < \kappa^{+\alpha}\rangle$ such that $\langle M_{\kappa^{+\alpha}}, \langle M_\gamma \mid \gamma < \kappa^{+\alpha}\rangle\rangle \models \mathsf{LCC}(\alpha)$ for all $\alpha \in (\kappa, \kappa^{+\alpha})$ with $|\alpha| \in \mathrm{Reg}$.

The main result of this section is the following:

Theorem C. *Suppose V models $\mathsf{ZFC} + \mathsf{GCH}$ and κ is a regular cardinal and β is an ordinal. Then there exists a set-sized forcing $\mathbb{P}$ which is cardinal preserving, cofinality preserving, GCH preserving and forces the existence of a filtration $\vec{M}$ such that $\Psi(\vec{M}, \kappa, \kappa^{+\beta})$ holds and $\langle M_{\kappa^{+\beta}}, \in, \vec{M}\rangle \models \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta})$.*

We start recalling the forcing from [HWW15] which we will iterate to obtain our model. We present the definitions of the forcing from [HWW15] for self containment, we will work mainly with abstract properties of the forcing from Theorem 3.10 below.

Convention 3.3.
- If a, b are sets of ordinals we write $a \triangleleft b$ iff $\sup(a) \cap b = a$.
- If $\mathbb{P} = \langle\langle \mathbb{P}_\alpha \mid \alpha < \beta\rangle, \langle \dot{\mathbb{Q}}_\alpha \mid \alpha + 1 < \beta\rangle\rangle$ is a forcing iteration, given $\zeta < \beta$ we denote by $\dot{\mathbb{R}}_{\zeta,\beta}$ a $\mathbb{P}_\alpha$-name such that $\mathbb{P} = \mathbb{P}_\zeta * \dot{\mathbb{R}}_{\zeta,\beta}$ (For the existence of such name $\dot{\mathbb{R}}_{\zeta,\beta}$ see for example [Bau83, Section 5]).

Definition 3.4. Let κ be a regular cardinal. Suppose $\kappa \leq \alpha < \kappa^+$, a *condition at* α is a pair (f_α, c_α) which is either trivial, i.e. $(f_\alpha, c_\alpha) = (\emptyset, \emptyset)$, or there is $\gamma_\alpha < \kappa$ such that

(1) $c_\alpha : \gamma_\alpha \to 2$ is such that $C_\alpha((f_\alpha, c_\alpha) := \{\delta < \gamma_\alpha \mid c_\alpha(\delta) = 1\} = c_\alpha^{-1}\{1\}$.
(2) $f : \max(C_\alpha) \to \alpha$ is an injection and
(3) $f_\alpha[\max(C_\alpha)] \subseteq \max(C_\alpha)$

Definition 3.5. Let κ be a regular cardinal and $\alpha \in (\kappa, \kappa^+)$ we also define a function A with domain $[\kappa, \kappa^+)$ such that for every α, $A(\alpha)$ is a $\mathbb{H}_\alpha$-name for either 0 or 1. We fix a wellorder $\mathcal{W}$ of H_{κ^+} of order-type κ^+. Let $\beta \in [\kappa, \kappa^+)$ and assume that $A \restriction \beta$ and $\mathbb{H}_\beta$ have been defined.

Let $A(\beta)$ be the canonical $\mathbb{H}_\beta$-name for either 0 or 1 such that for any $\mathbb{H}_\beta$-generic G_β, $A(\beta) = 1$ iff $\beta = \prec\gamma, \prec\delta, \varepsilon\succ\succ$ [6], $\dot{x}$ is the γ^{th}-name (in the sense of $\mathcal{W}$) $\mathbb{H}_\delta$-nice name for a subset of κ, $\varepsilon < \kappa$ and $\varepsilon \in \dot{x}^{G_\beta}$ [7]

Suppose that $A\restriction\beta$ is defined, we proceed to define $\mathbb{H}_\beta$. Suppose p is an β-sequence such that for each $\alpha < \beta$ we have $p(\alpha) \in \mathbb{H}_\alpha$ and suppose that $|\operatorname{supp}(p)| = |\{\tau < \beta \mid p(\tau) \neq 1_{\mathbb{H}_\tau}\}| < \kappa$. If $\beta = \alpha + 1$ for some α, then we require that

- $p \restriction \beta \in \mathbb{H}_\beta$ for every $\beta < \alpha$ and if $\alpha = \beta + 1$, the following holds:
- $p(\beta) = (f_\beta, c_\beta)$ is a condition at β,
- if $C_\beta \neq \emptyset$, then $p \restriction \beta$ decides $A(\beta) = a_\beta$,
- for all $\delta \in C_\beta \neq \emptyset$,$p(\operatorname{otp}(f_\beta[\delta])) = a_\beta$,
- $\gamma^p = \operatorname{supp}(p) \cap \kappa = \gamma_\beta = dom(c_\beta)$ for any $\beta \in C\text{-}\operatorname{supp}(p)$, where $C\text{-}\operatorname{supp}(p) := \{\gamma < \beta \mid C_\gamma(p(\gamma) \neq \emptyset\}$
- $\exists \delta^p$ such that for all $\beta \in C\text{-}\operatorname{supp}(p), \max(C_\beta) = \delta^p$,
- $\beta_0 < \beta_1$ both in $C\text{-}\operatorname{supp}(p)$,
$$f_{\beta_0}[\delta^p] \lhd f_{\beta_1}[\delta^p]$$
[8] and
$$f_{\beta_1}[\delta^p] \setminus \beta_0 \neq \emptyset$$
For p and q in $\mathbb{H}_\alpha$ we let $q \leq p$ iff $q \restriction \kappa \leq p \restriction \kappa$ and for every $\beta \in [\lambda, \alpha)$, $q(\beta) \leq p(\beta)$.

We will work with a forcing that is equivalent to $\mathbb{H}_\beta$ and is a subset of H_{κ^+}.

Definition 3.6. If κ is a regular cardinal and $\pi : \mathbb{H}_{\kappa,\kappa^+} \to H_{\kappa^+}$ is such that $\pi(p) = p \restriction \operatorname{supp}(p)$, then we define $\mathbb{P}_{\kappa,\kappa^+} := \operatorname{rng}(\pi)$ and given $s, t \in \mathbb{P}_{\kappa,\kappa^+}$ we let $s \leq_{\mathbb{P}_{\kappa,\kappa^+}} t$ iff $\pi^{-1}(s) \leq_{\mathbb{H}_{\kappa,\kappa^+}} \pi^{-1}(t)$.

Next we describe how we will iterate the forcing from Definition .

Definition 3.7. Let α be an ordinal and κ a regular cardinal. We define $\mathbb{P}_{\kappa,\kappa^{+\alpha}}$ as the iteration $\langle\langle \mathbb{P}_{\kappa,\kappa^{+\tau}} \mid \tau \leq \alpha\rangle, \langle \dot{\mathbb{Q}}_\tau \mid \tau < \alpha\rangle\rangle$ as follows:

(1) If $\tau = \beta + 1$ for some $\beta < \tau$ and $\kappa^{+\beta}$ is a regular cardinal. If there exists $\dot{\mathbb{Q}}_\beta \subseteq H_{\kappa^{+\beta+1}}$ such that $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash \dot{\mathbb{Q}}_\beta = \mathbb{P}_{\kappa^{+\beta},\kappa^{+\beta+1}}$ we let $\mathbb{P}_{\kappa,\kappa^{+\beta+1}} = \mathbb{P}_{\kappa,\kappa^{+\beta}} * \dot{\mathbb{Q}}_\beta$, otherwise we stop the iteration.
(2) If $\tau = \beta + 1$ for some $\beta < \tau$ and $\kappa^{+\beta}$ is a singular cardinal, we let $\dot{\mathbb{Q}}_\beta = \check{1}$.
(3) If β is a limit ordinal and $\kappa^{+\beta}$ is a regular cardinal, then $\mathbb{P}_\beta$ is the direct limit of $\langle \mathbb{P}_{\kappa,\kappa^{+\beta}}, \dot{\mathbb{Q}}_\beta \mid \theta < \tau\rangle$.
(4) If τ is a limit ordinal and $\kappa^{+\beta}$ is singular, then $\mathbb{P}_{\kappa,\kappa^{+\tau}}$ is the inverse limit of $\langle \mathbb{P}_{\kappa,\kappa^{+\theta}}, \dot{\mathbb{Q}}_\theta \mid \theta < \tau\rangle$.

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[6] $\prec\gamma, \prec\delta, \varepsilon\succ\succ$ denotes the Gödel pairing.
[7] As $\delta < \beta$, we identify $\dot{x}$ with a $\mathbb{H}_\beta$-name using the induction hypothesis that $\mathbb{H}_\delta \prec \mathbb{H}_\beta$.
[8] See Convention 3.3

Remark 3.8. Given an ordinal α and a regular cardinal κ the forcing $\mathbb{P}_{\kappa,\kappa^{+\alpha}}$ is obtained forcing $\mathbb{P}_{\kappa^\beta,\kappa^{+\beta+1}}$ for each successor ordinal $\beta < \alpha$ such that $\kappa^{+\beta}$ is a regular cardinal. If β is a limit ordinal but not an inaccessible cardinal, then we take inverse limits, if κ is an inaccessible cardinal we take direct limits.

Remark 3.9. Let κ be a regular cardinal and let G be $\mathbb{P}_{\kappa,\kappa^+}$-generic. Consider

$$f_\alpha := \bigcup\{f \mid \exists p(\alpha \in \mathrm{dom}(p) \wedge p \in G \wedge p(\alpha) = (c, f))\}$$

By a standard density argument we have that f_α is a bijection from κ onto α. It also holds that $\alpha \in A$ iff $\{\gamma < \kappa \mid \mathrm{otp}(f_\alpha[\gamma]) \in A\}$ contains a club and $\alpha \notin A$ iff $\{\gamma < \kappa \mid \mathrm{otp}(f_\alpha[\gamma]) \notin A\}$ contains a club.

Theorem 3.10 ([HWW15]). *Suppose* GCH *holds and κ is a cardinal. Then $\mathbb{P}_\kappa$ is a $<\kappa$- directed closed, κ^+-cc forcing such that $|\mathbb{P}| = \kappa$ and for all G, $\mathbb{P}$-generic, the following holds in $V[G]$:*

- *There is $\vec{M} = \langle M_\alpha \mid \alpha \leq \kappa^+\rangle$ which witnesses $\mathsf{LCC}(\kappa, \kappa^+)$,*
- *$M_\kappa = H_\kappa$,*
- *$M_{\kappa^+} = H_{\kappa^+}$,*
- *There exists $A \subseteq \kappa^+$ such that for all $\beta < \kappa^+$ we have $(M_\beta = L_\beta[A])$.*

We will need the following facts:

Fact 3.11. [Bau83, Theorem 2.7] *Let $\mathbb{P}_\alpha$ be the inverse limit of $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha\rangle$. Suppose that κ is a regular cardinal and for all $\beta < \alpha$,*

$$\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta \text{ is } \kappa\text{-directed closed.}$$

Suppose also that all limits are inverse or direct and that if $\beta \leq \alpha$, β is a limit ordinal and $\mathrm{cf}(\beta) < \kappa$, then $\mathbb{P}_\beta$ is the inverse limit of $\langle \mathbb{P}_\gamma \mid \gamma < \beta\rangle$. Then $\mathbb{P}_\alpha$ is κ-directed closed.

Fact 3.12. *Suppose $\mathrm{cf}(\kappa) > \omega$. If $\mathbb{P}$ is a κ-cc forcing and $\mathbb{P} \Vdash \dot{\mathbb{Q}}$ is κ-cc, then $\mathbb{P} * \mathbb{Q}$ is κ-cc.*

Fact 3.13. *Let $\mathbb{P}$ be a partial order and θ a regular cardinal. Suppose $\mathbb{P}$ preserves cardinals. If G is $\mathbb{P}$-generic, then $H_\theta^V = H_\theta^{V[G]}$.*

Proof. Let G be $\mathbb{P}$-generic and $w \in H_\theta^{V[G]}$. Let $\delta = |\mathrm{trcl}\{w\}|$ and suppose $f : \delta \to \mathrm{trcl}\{w\}$ is a bijection. Let $R \subseteq \delta$ such that $(x, y) \in R$ if and only if $f(x) \in f(y)$. Then the Mostowski collapse of (δ, R) is equal to w. Since $\mathbb{P}$ is $<\theta$-closed, it follows that $(\delta, R) \in V$ and hence $w \in V$. Since $\mathbb{P}$ preserves cardinals it follows that $|w| = \delta$ and $w \in H_\theta$. □

Fact 3.14. *Let μ be a cardinal. Suppose $\mathbb{P}$ is a forcing that is μ^+-cc and $\mathbb{P} \subseteq H_{\mu^+}$. If G is $\mathbb{P}$-generic, then $H_{\mu^+}^{V[G]} = H_{\mu^+}[G]$.*

Proof. We proceed by $\in$ induction on the elements of $H_{\mu^+}^{V[G]}$. Notice that it suffices to prove the result for subsets of κ^+, since every $x \in H_{\kappa^+}^{V[G]}$ is of the form $\mathrm{trcl}(\gamma, R)$ for some $\gamma < \kappa^+$ and $R \subseteq \gamma \times \gamma$. Let $x = \sigma[G] \in H_{\mu^+}^{V[G]}$ such that $x \subseteq \kappa^+$. As $\mathbb{P}$ is μ^+-cc, there is an ordinal γ such that $x \subseteq \gamma$ and $1_{\mathbb{P}} \Vdash \sigma \subseteq \gamma$. Let $\theta = \bigcup\{A_\tau \times \{\tau\} \mid \exists p \in \mathbb{P} \exists \xi \in \gamma (p \Vdash \check{\xi} = \pi) \wedge \tau \in H_{\mu^+}\}$ and each $A_\tau \subseteq \mathbb{P}$ such that:

(1) A_τ is an antichain,
(2) $q \in A_\tau$ implies $q \Vdash \tau \in \sigma$,
(3) $q \in A_\tau$ is maximal with respect to the above two properties.

It follows that $\theta \in H_{\mu^+}$ and $\theta[G] = \sigma[G]$. □

Remark 3.15. If $\mathbb{P}$ and μ satisfy the hypothesis from Fact 3.14 and σ is a $\mathbb{P}$-name such that $1_{\mathbb{P}} \Vdash \sigma \subseteq H_{\mu^+}$, then using Fact 3.14 we can find a $\mathbb{P}$-name $\pi \subseteq H_{\mu^+}$ such that $1_{\mathbb{P}} \Vdash \sigma = \pi$.

Remark 3.16. Given a regular cardinal κ and a limit ordinal β, we have $\mathrm{cf}(\kappa^{+\beta}) = \mathrm{cf}(\beta)$. Therefore if $\beta < \kappa^{+\beta}$ it follows that $\kappa^{+\beta}$ is singular. On the other hand if $\kappa^{+\beta} = \mathrm{cf}(\kappa^{+\beta}) = \mathrm{cf}(\beta) \leq \beta$, then β is a weakly inaccessible cardinal, i.e. a cardinal that is a limit cardinal and regular. Thus $\kappa^{+\beta}$ is regular iff β is a weakly inaccessible cardinal.

Convention 3.17. Let $\mathbb{P}$ be a set forcing and $\varphi(\sigma_0, \cdots, \sigma_n)$ a formula in the forcing language. We write $\mathbb{P} \Vdash \varphi(\sigma_0, \cdots, \sigma_n)$ if for all $p \in \mathbb{P}$ we have $p \Vdash \varphi(\sigma_0, \cdots, \sigma_n)$.

Lemma 3.18. *Suppose* GCH *holds. Let* κ *be a regular cardinal and* β *an ordinal. Then* $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ *preserves* GCH*, cardinals and cofinalities and if* $\kappa^{+\beta}$ *is a regular cardinal then there exists* $\dot{\mathbb{Q}}_\beta \subseteq H_{\kappa^{+\beta+1}}$ *a* $\mathbb{P}_{\kappa,\kappa^{+\beta}}$*-name such that* $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash \mathbb{P}_{\kappa^{+\beta},\kappa^{+\beta+1}} = \dot{\mathbb{Q}}_\beta$.

Proof. We prove the lemma by induction. Besides the statement of the lemma we carry the following additional induction hypothesis:

$(1)_\beta$ $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ preserves cardinals and cofinalities,

$(2)_\beta$ If $\kappa^{+\beta}$ is a regular cardinal and not the successor of a singular cardinal, then $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ is $\kappa^{+\beta}$-cc and there exists $\dot{\mathbb{Q}}_\beta \subseteq H_{\kappa^{+\beta+1}}$ such that $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash (\mathbb{P}_{\kappa^{+\beta},\kappa^{+\beta+1}} = \dot{\mathbb{Q}}_\beta)$.

$(3)_\beta$ If $\kappa^{+\beta}$ is a successor of a singular cardinal, then $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ is $\kappa^{+\beta+1}$-cc,

$(4)_\beta$ If $\kappa^{+\beta}$ is a singular cardinal, then $|\mathbb{P}_\beta| \leq \kappa^{+\beta+1}$

For $\beta = 1$ the lemma follows from Theorem 3.10. Suppose that $(1)_\theta$ to $(4)_\theta$ and that the lemma holds for all $\theta < \beta$. We will verify that $(1)_\beta$ to $(4)_\beta$ and that the lemma holds for β.

► Suppose $\beta = \theta + 1$ for some ordinal θ such that $\kappa^{+\theta}$ is regular. From $(2)_\theta$ in our induction hypothesis, $\mathbb{P}_{\kappa,\kappa^{+\theta}}$ is $\kappa^{+\theta}$-cc, hence by Fact 3.14, for any G, $\mathbb{P}_{\kappa,\kappa^{+\theta}}$-generic, we have $H_{\kappa^{+\theta+1}}[G] = H^{V[G]}_{\kappa^{+\theta+1}}$. Thus there exists $\dot{\mathbb{Q}}_\beta \subseteq H_{\kappa^{\theta+1}}$ such that $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash$ "$\mathbb{P}_{\kappa^\beta,\kappa^{+\beta+1}} = \dot{\mathbb{Q}}_\beta$".

From our induction hypothesis $\mathbb{P}_{\kappa,\kappa^{+\theta}}$ preserves GCH, cardinals and cofinalities and from $(2)_\theta$ we have that $\mathbb{P}_{\kappa,\kappa^{+\theta}}$ is $\kappa^{+\theta}$-cc. We also have that

$$\mathbb{P}_{\kappa,\kappa^{+\theta}} \Vdash \text{“}\mathbb{P}_{\kappa^{+\theta},\kappa^{+\theta+1}} \text{ preserves } \mathsf{GCH}, \text{ cardinals, cofinalities and it is } \kappa^{+\theta+1}\text{-cc”}.$$

Altogether implies that $\mathbb{P}_{\kappa,\kappa^{+\beta}}$, which is $\mathbb{P}_{\kappa,\kappa^{+\theta}} * \dot{\mathbb{Q}}_\beta$, preserves GCH, cardinals and cofinalities. By Fact 3.12 we have that $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ is $\kappa^{+\beta}$-cc.

► Suppose $\kappa^{+\theta}$ is singular and $\beta = \theta + 1$. By our induction hypothesis $(4)_\theta$ we have $|\mathbb{P}_{\kappa^{+\theta}}| \leq \kappa^{+\theta+1}$. As $\dot{\mathbb{Q}}_\theta$ is the trivial forcing, it follows that $|\mathbb{P}_{\kappa,\kappa^{+\beta}}| \leq \kappa^{+\beta}$ and $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ is $\kappa^{+\beta+1}$-cc. Therefore if G is $\mathbb{P}_{\kappa,\kappa^{+\beta+1}}$-generic, $H_{\kappa^{+\beta+1}}[G] = H^{V[G]}_{\kappa^{+\beta+1}}$, hence we can find $\dot{\mathbb{Q}}_{\beta+1} \subseteq H_{\kappa^{+\beta+1}}$ as sought and $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ preserves GCH, cardinals and cofinalities.

► Suppose that β is a limit ordinal and $\kappa^{+\beta}$ is a singular cardinal.

From our induction hypothesis we have that for every $\zeta < \beta$ the forcing $\mathbb{P}_{\kappa,\kappa^{+\zeta}}$ preserves cardinals and by Theorem 3.11

$$\mathbb{P}_{\kappa,\kappa^{+\zeta}} \Vdash \text{“}\dot{\mathbb{R}}_{\kappa^{+\zeta},\kappa^{+\beta}} \text{ is } {<}\kappa^{+\tau}\text{-closed.”}$$

Therefore all cardinals below $\kappa^{+\beta}$ are preserved. Thus $\kappa^{+\beta}$ remains a cardinal in $V[G_\beta]$ and $\mathrm{cf}(\kappa^{+\beta})^{V[G_\beta]} = (\mathrm{cf}(\kappa^{+\beta}))^{V[G_\tau]} = (\mathrm{cf}(\kappa^{+\beta}))^V$.

As $\mathrm{cf}(\kappa^{+\beta})^+ < \kappa^{+\beta}$, we can fix $\tau < \beta$ such that $\kappa^{+\tau} \geq cf(\kappa^{+\beta})$. From our induction hypothesis we have that $\mathbb{P}_{\kappa,\kappa^{+\tau+1}}$ preserves cardinals. From Theorem

3.11 we have that $\mathbb{P}_{\kappa,\kappa^{+\tau+1}}$ forces $\dot{\mathbb{R}}_{\kappa^{+\tau+1},\kappa^{+\beta}}$ to be $<\mathrm{cf}(\kappa^{+\beta})^+$-closed. Therefore $((\kappa^{+\beta})^{\mathrm{cf}(\kappa^{+\beta})})^{V[G_\tau]} = ((\kappa^{+\beta})^{\mathrm{cf}(\kappa^{+\beta})})^{V[G_\beta]}$ and $(\kappa^{+\beta+1})^{V[G_\theta]} = (\kappa^{+\beta+1})^{V[G_\beta]}$. We have verified above that

- $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash (\kappa^{+\beta})^V$is a cardinal
- $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash (\mathrm{cf}(\kappa^{+\beta})) = \mathrm{cf}^V(\kappa^{+\beta})$
- $\mathbb{P}_{\kappa,\kappa^{+\beta}} \Vdash 2^{\kappa^{+\beta}} = \kappa^{+\beta+1}$

It is also clear from the above that $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ preserves GCH, cardinals and cofinalities below $\kappa^{+\beta}$.

From our induction hypothesis $(2)_\theta$ it follows that for each $\theta < \beta$ we have $|\dot{\mathbb{Q}}_\theta| \leq \kappa^{+\theta+1}$, then using GCH it follows that $|\mathbb{P}_{\kappa,\kappa^{+\beta}}| \leq \kappa^{+\beta+1}$ and hence $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ is $\kappa^{+\beta+2}$-cc.

Thus $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ preserves GCH, cardinals and cofinalities above $\kappa^{+\beta+2}$.

▶ If $\kappa^{+\beta}$ is a limit cardinal and regular, then $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ is the direct limit of $\langle \mathbb{P}_{\kappa,\kappa^{+\tau}}, \dot{\mathbb{Q}}_{\tau+1} \mid \tau < \kappa^{+\beta}\rangle$. From our induction hypothesis $(2)_\theta$ for $\theta < \beta$, we have $|\dot{\mathbb{Q}}_\tau| \leq \kappa^{+\tau+1}$. Therefore $|\mathbb{P}_{\kappa^{+\beta}}| \leq \kappa^{+\beta}$ and hence $\mathbb{P}_{\kappa^{+\beta}}$ is $\kappa^{+\beta+1}$-cc and preserves GCH, cardinals and cofinalties at cardinals greater or equal than $\kappa^{+\beta}$. From our induction hypothesis we have that cofinalities cardinals and GCH are preserved below $\kappa^{+\beta}$. Hence $\mathbb{P}_{\kappa,\kappa^{+\beta}}$ preserves cofinalities, cardinals and GCH. □

Lemma 3.19, below, will be used in a context where $W_\tau = V[G_\tau]$ and G_τ is $\mathbb{P}_{\kappa,\kappa^{+\tau}}$-generic.

Lemma 3.19. *Let $\langle W_\tau \mid \tau \leq \beta\rangle$ be a sequence of transitive proper classes that model* ZFC *and suppose that $\tau_0 < \tau_1 < \beta$ implies $W_{\tau_0} \subseteq W_{\tau_1}$ and $\mathrm{Card}^{W_{\tau_0}} = \mathrm{Card}^{W_{\tau_1}}$. Suppose further that the following hold:*

(1) for each $\tau < \beta$ the folowing holds in W_τ: there exists $A_\tau \subseteq \kappa^{+\tau}$ such that $\vec{M}^\tau = \langle L[A_\tau]_\zeta \mid \zeta < \kappa^{+\tau+1}\rangle$ witnesses $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\tau})$,

(2) For $\tau_0 < \tau_1 < \beta$ we have $H^{W_{\tau_0}}_{\tau_0^+} = L[A_{\tau_0}]_{\tau_0^+} = L[A_{\tau_1}]_{\tau_1^+} = H^{W_{\tau_1}}_{\tau_1^+}$,

(3) for every $\tau < \beta$ we have $H^{W_\tau}_\tau = H^{W_\beta}_\tau$ and

(4)

$$\mathbb{A} := \bigcup\{A_\tau \restriction (\kappa^{+\tau}, \kappa^{+\tau+1}) \mid \mathrm{Reg}(\kappa^{+\tau}) \wedge \tau < \beta\} \cup \bigcup\{A_\tau \restriction (\kappa^{+\tau}, \kappa^{+\tau+2}) \mid Sing(\kappa^{+\tau})\}$$

is an element of W_β.

Then $\vec{M} = \langle L_\zeta[\mathbb{A}] \mid \zeta < \kappa^{+\beta}\rangle$ witnesses $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta})$ in W_β.

Proof. We work in W_β. Let $\alpha \in (\kappa, \kappa^{+\beta})$ such that $|\alpha|$ is a regular cardinal. Let $\mathbb{S} = \langle L_\alpha[\mathbb{A}], \in, (\mathcal{F}_n)_{n\in\omega}\rangle \in H_{|\alpha|^+}$.

We will find $\vec{B}$ that witnesses LCC at α for $\mathbb{S}$. There is $\vec{B_0} \in W_\tau$ where $\kappa^{+\tau} = |\alpha|$, which witnesses LCC at α in W_τ with respect to $(\mathcal{F}_n)_{n\in\omega}$. Since $L_\tau[A_\tau] = H^{W_\tau}_\tau = H^{W_{\tau^+}}_\tau = L_\tau[A_{\tau^+}]$, it follows that there is a club $C \subseteq \kappa^{+\tau}$ such that $\iota \in C$ implies $L_\iota[A_\tau] = L_\iota[\mathbb{A}]$. Thus $\vec{B} = \vec{B_0} \restriction C$ will witness LCC at α with respect to $(\mathcal{F}_n)_{n\in\omega}$ in W_β. □

Lemma 3.20. *Let κ be a regular cardinal and β an ordinal. Suppose that there exists $\vec{M} = \langle L_\alpha[A] \mid \kappa \leq \alpha < \kappa^{+\beta}\rangle$ which witnesses $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta})$ and $\Psi(\vec{M}, \kappa, \kappa^{+\beta})$ holds. If $\kappa^{+\beta}$ is a regular cardinal, then $\mathbb{P}_{\kappa^{+\beta},\kappa^{+\beta+1}} \Vdash \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta})$ and if $\kappa^{+\beta}$ is a singular cardinal then $\mathbb{P}_{\kappa^{+\beta+1},\kappa^{+\beta+2}} \Vdash \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta+1}) \wedge \Psi(\vec{M}, \kappa, \kappa^{+\beta})$.*

Proof. We split the proof into two cases depending on whether $\kappa^{+\beta}$ is regular or not.

▶ Suppose $\kappa^{+\beta}$ is a regular cardinal. Let $B \subseteq (\kappa^\beta, \kappa^{\beta+1})$ such that $\langle L_\alpha[B] \mid \alpha < \kappa^{+\beta+1}\rangle$ witnesses $\mathsf{LCC}_{\mathrm{Reg}}(\kappa^\beta, \kappa^{+\beta+1})$. Since $\mathbb{P}_{\kappa^{+\beta},\kappa^{+\beta+1}}$ is $<\kappa^{+\beta}$-closed, it follows that for G, $\mathbb{P}_{\kappa^{+\beta},\kappa^{+\beta+1}}$-generic we have, by Fact 3.13 that $(H_{\kappa^{+\beta}})^{V[G]} = (H_{\kappa^{+\beta}})^V$. We then let $\vec{N} = \langle L_\alpha[C] \mid \alpha < \kappa^{+\beta+1}\rangle$ where $C = (A \cap \kappa^{+\beta}) \cup (B \setminus \kappa^{+\beta})$, witness $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta+1})$.

▶ Suppose $\kappa^{+\beta}$ is a singular cardinal. Let G be $\mathbb{P}_{\kappa^{+\beta+1},\kappa^{+\beta+2}}$-generic over V. Let G be $\mathbb{P}$-generic, from Fact 3.13 it follows that for every cardinal $\theta < \kappa^{+\beta+1}$ we have $H^V_\theta = H^{V[G]}_\theta$.

Let $B \subseteq \kappa^{+\beta+2}$ be such that $\vec{N} = \langle L_\gamma[B] \mid \gamma < \kappa^{+\beta+2}\rangle$ witnesses $\mathsf{LCC}(\kappa^{+\beta+1}, \kappa^{+\beta+2})$ in $V[G]$. Let $C := A \cup (B \setminus \kappa^{+\beta})$. Then $\vec{W} := \langle L_\alpha[C] \mid \alpha < \kappa^{+\beta+2}\rangle$ witnesses $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta+2})$. □

Theorem C. *If* GCH *holds and κ is a regular cardinal and α is an ordinal, then there is a set forcing $\mathbb{P}$ which is $< \kappa$-directed closed and $\kappa^{+\alpha+1}$-cc,* GCH *preserving such that in $V^{\mathbb{P}}$ there is a filtration $\langle M_\alpha \mid \alpha < \kappa^{+\alpha}\rangle$ such that $\Psi(\vec{M}, \kappa, \kappa^{+\beta+1})$ holds and $\langle M_\alpha \mid \alpha < \kappa^{+\alpha}\rangle \models \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\alpha})$*

Proof. We prove by induction that the following hold:

(1) for each $\tau < \beta$ there exists $A_\tau \subseteq \kappa^{+\tau} \in V[G_\tau]$ such that, in $V[G_\tau]$ we have that $\vec{M}^\tau = \langle L[A_\tau]_\zeta \mid \zeta < \kappa^{+\tau+1}\rangle \models \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\tau})$ and $\Psi(\vec{M}, \kappa, \kappa^{+\tau})$
(2) For $\tau_0 < \tau_1 < \beta$ we have $H^{V[G_\tau]}_{\tau_0^+} = L_{\tau_0^+}[A_{\tau_0}] = L_{\tau_0^+}[A_{\tau_1}] = H^{V[G_\tau]}_{\tau_1^+}$,
(3) for every $\tau < \beta$ we have $H^{V[G_\tau]}_\tau = H^{V[G_\beta]}_\tau$ and
(4)

$$\mathbb{A} := \bigcup\{A_\tau \restriction (\kappa^{+\tau}, \kappa^{+\tau+1}) \mid \mathrm{Reg}(\kappa^{+\tau}) \wedge \tau < \beta\} \cup \bigcup\{A_\tau \restriction (\kappa^{+\tau}, \kappa^{+\tau+2}) \mid \mathrm{Sing}(\kappa^{+\tau})\}$$

is an element of $V[G_\beta]$.

If $\beta = 1$ the lemma follows from Theorem 3.10. If $\beta = \theta + 1$, from our induction hypothesis and Lemma 3.20 it follows that $\mathbb{P}_{\kappa,\kappa^{+\beta+}} \Vdash \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\beta})$. If β is a limit ordinal all we need to verify is that

$$H^{V_\tau}_{\kappa^{+\tau+1}} = H^{V_\beta}_{\kappa^{+\tau+1}} \tag{1}$$

for every $\tau < \beta$ in order to apply Lemma 3.19. Since for each $\tau < \beta$ we have that $\mathbb{P}_{\kappa,\kappa^{+\zeta}} \Vdash$ "$\dot{\mathbb{R}}_{\kappa^{+\zeta}}$ is $< \kappa^{+\zeta}$-closed" and $\mathbb{P}_{\kappa,\kappa^{+\zeta}}$ preserve cardinals and cofinalities (1) follows from Fact 3.13

□

4. Applciations

In this section we show that the iteration of the forcing from [HWW15] can replace some uses of the main forcing in [FH11].

Definition 4.1. Let $\mu, A, \vec{F}$ be sets. We say that $\Xi(A, \mu, \vec{f})$ holds iff μ is a regular cardinal, A is a function such that $A : \mu^+ \to 2$ and $\vec{f}$ is a sequence of bijections $\langle f_\beta \mid \kappa \leq \beta < \mu^+\rangle$ such that for each $\beta < \mu$, $f_\beta : \mu \to \beta$, and the following hold:

- $H_{\mu^+} = L_{\mu^+}[A]$,
- $(\xi, 1) \in A \setminus \mu \leftrightarrow \exists C(C \text{ is a club } \wedge C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\xi[\gamma]) \in A\})$,
- $(\xi, 1) \in A \setminus \mu \leftrightarrow \exists C(C \text{ is a club } \wedge C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\xi[\gamma]) \in A\})$.

Lemma 4.2. *Let μ be a regular cardinal, A a function $A : \mu^+ \to 2$ and $\vec{f} = \langle f_\beta \mid \kappa \leq \beta < \mu^+\rangle$ a sequence of bijections such that $f_\beta : \mu \to \beta$ for each $\beta < \mu^+$. Suppose $\Xi(A, \mu, \vec{f})$ holds. Given $\zeta \in \mu^+ \setminus \mu$, the following are equivalent:*

(1) $\zeta \in A \setminus \mu$,
(2) $\exists f \exists C(f : \mu \to \zeta \wedge f$ *is a bijection* $\wedge C$ *is a club* $\wedge C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f[\gamma]) \in A\}$,

(3) $\forall f \exists C(f : \mu \to \zeta \wedge f$ *is a bijection* $\wedge C$ *is a club* $\wedge C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f[\gamma]) \in A\}$,

(4) $\forall f \forall C(f : \mu \to \zeta \wedge f$ *is a bijection* $\wedge C$ *is a club* $\to C \nsubseteq \{\gamma < \mu \mid \mathrm{otp}(f[\gamma]) \notin A\}$.

Moreover $\xi \in A \setminus \mu$ *is* $\Delta_1(\{A \restriction \mu, \xi\})$ *over* H_{μ^+}.

Proof. Let $\zeta \in \mu^+ \setminus \mu$. As $\mu, A, \vec{f}$ witness the condensation axiom, it follows that $\zeta \in A \setminus \mu \leftrightarrow \exists C(C$ is a club $\wedge C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\zeta[\gamma]) \in A\}$. Let f be a bijection from μ onto ζ. Then from the regularity of μ it follows that there exists (C a club such that $D \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\zeta[\gamma]) \in A\}$ iff there exists D a club such that $C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\zeta[\gamma]) \in A\}$. Thus (1) (2) and (3) are equivalent.

Let us verify that (4) is equivalent to (1). Since $\mu, A, \vec{f}$ witness the condensation axiom, it follows that $\zeta \notin A$ iff there exists a club C such that $C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\zeta[\gamma]) \notin A\}$. Let f be a bijection $f : \mu \to \zeta$. From the regularity of μ it follows that there exists a club C such that $C \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\zeta[\gamma]) \notin A\}$ iff there exists a club D such that $D \subseteq \{\gamma < \mu \mid \mathrm{otp}(f_\zeta[\gamma]) \notin A\}$. Thus (1) is equivalent to (4).

The moreover part follows from the equivalence between (1),(2) and (4), and the fact that $C \nsubseteq \{\gamma < \mu \mid \mathrm{otp}(f[\gamma]) \notin A\}$ is equivalent to $\forall h \forall \gamma \forall \beta((\gamma \in C \wedge h : \beta \to f[\gamma] \wedge h$ is an isomorphis$) \to (\beta, 0) \in A)$. ☐

Our next result, Theorem D, is an adaptation of [FH11, Theorem 39].

Theorem D. *Suppose that* θ *is an ordinal,* κ *is a regular cardinal and* $\kappa^{+\theta}$ *is a regular cardinal. Then* $\mathbb{P}_{\kappa,\kappa^{+\theta+1}}$ *forces that* $\mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\theta+1})$ *holds and that there exists a well order of* H_{κ^+} *that is* Δ_1 *definable over* H_{κ^+} *in a parameter* $a \subseteq \kappa^{+\theta}$.

Proof. We have that $\mathbb{P}_{\kappa,\kappa^{+\theta+1}}$ forces that there exists $\vec{M} = \langle M_\alpha \mid \alpha < \kappa^{+\theta+1}\rangle$, a filtration, such that

(1) $H_{\kappa^{+\beta}} = M_{\kappa^{+\beta}}$ for every $\beta \leq \theta + 1$,
(2) there exists $A \subseteq \kappa^{+\theta+1}$ such that for all $\alpha < \kappa^{+\theta+1}$ we have $M_\alpha = L_\alpha[A]$
(3) $\langle M, \in, \vec{M}\rangle \models \mathsf{LCC}_{\mathrm{Reg}}(\kappa, \kappa^{+\theta+1})$

Let $\langle f_\beta \mid \kappa < \beta < \kappa^+\rangle$ be the sequence of bijections obtained by forcing with $\mathbb{P}_{\kappa,\kappa^+}$, see remark 3.9. Then we have that $\beta \in A$ iff $\{\gamma < \kappa^{+\theta} \mid \mathrm{otp}(f_\beta[\delta]) \in A\}$ contains a club and $\beta \notin A$ iff $\{\gamma < \kappa^{+\theta} \mid \mathrm{otp}(f_\beta[\gamma]) \notin A\}$ contains a club.

Therefore by Lemma 4.2 we can define $A \cap \kappa^{+\theta+1}$ in $H_{\kappa^{+\theta+1}}$ using $A \cap \kappa^{+\theta}$ with a Δ_1 formula. The concatenation of the definition of A with the Δ_1 well order of $L_{\kappa^{+\theta+1}}[A]$ gives the Δ_1 well order we sought.

☐

Corollary D. *Suppose that* θ *is an ordinal,* κ *is a regular cardinal. Then* $\mathbb{P}_{\kappa,\kappa^{+\theta+1}}$ *forces that for every* $S \subseteq \kappa$ *stationary we have* $\mathrm{Dl}^*_S(\Pi^1_2)$ *and in particular* $\diamondsuit(S)$.

Proof. Follows from Theorem D and [FMR20, Theorem 2.24]. ☐

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Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.
URL: http://u.math.biu.ac.il/~zanettg
Email address: zanettg@macs.biu.ac.il